B.Sc., STATISTICS

II Year

2.3 STATISTICAL DISTRIBUTIONS

II Year – 2.3 STATISTICAL DISTRIBUTIONS

Unit - I

Distribution functions of one dimensional and two dimensional random variables – applications of Jacobian, marginal, conditional distributions - expectation.

Unit - II

Discrete distributions: One-point distribution, Bernoulli, Binomial, Poisson, Recurrence relations for probabilities, Geometric and Negative binomial distributions – Hyper geometric distribution, Multinomial distribution and discrete Uniform distribution- Moments – moment generating function, Characteristic function, Cumulant Generating function. Fitting of Binomial and Poisson distributions.

Unit - III

Continuous distributions: Uniform, Normal, Cauchy and Lognormal distributions- concepts, moments, moment generating and characteristic functions and their properties.

Unit - IV

Exponential, Gamma, Beta (first and second kinds) concepts, moments, moment generating and characteristic functions and their properties.

Unit - V

Sampling distributions: Chi-square, t and F distributions- concepts, moments, moment generating and characteristic functions and their properties.

BOOKS FOR STUDY:

1. Gupta, S. C., and V. K. Kapoor (2000) Fundamentals of Mathematical Statistics, A Modern Approach (Eighth Edition). Sultan Chand & sons. New Delhi.
2. Alexander, M. Mood, Franklin A. Graybill and Duane C. Boes (1974) Introduction to the Theory of Statistics (Third Edition), Mc Graw Hill Comp Ltd. New Delhi.
3. Goon, A. M., M. K. Gupta and B. Dasgupta (2002) Fundamentals of Statistics, Vol. I, World Press Kolkata.
4. Rohatgi, V. K.and A. K. md. Ehsanes Saleh(2009) An Introduction to Probability Theory and Mathematical Statistics, 2^{nd} Edition, Wiley Eastern Limited, New Delhi.
5. Parimal Mukopadhyay (2006) Mathematical Statistics, (Third Edition), Books and Allied Private Limited, Kolkata.
6. Edward J. Dudewicz and Satya N. Mishra (2007) Modern Mathematical Statistics, John Wiley & Sons. Inc., New York.

Unit – I

Random Variables and Expectations

1.1 Introduction

In many experiments, we are interested not in knowing which of the outcomes has occurred, but in the numbers associated with them. For example, when n coins are tossed, one may be interested in knowing the number of heads obtained. When a pair of dice is tossed, one may seek information about the sum of points. Thus, we associate a real number with each outcome of an experiment. In other words, we are considering a function whose domain is the set of possible outcomes and whose range is a subset of the set of real numbers. Such a function is called a random variable (r.v).

Intuitively by a r.v, we mean a real number X connected with the outcome of a random experiment E. For example, if E consists of two tosses of a coin, we may consider the r.v which is the number of heads (0, 1 or 2).

Outcome	:	HH	HT	TH	TT
Value of X	:	2	1	1	0

Thus to each outcome ω, there corresponds a real number $X(\omega)$. Since the points of the sample space S correspond to outcomes, this means that a real number, which we denote by $X(\omega)$, is defined for each $\omega \in S$.

1.2 One dimensional Random Variable

Definition

A one dimensional r.v is a function that assigns a real number to each and every outcomes of a sample space.

There are two types of r.vs,

(i) Discrete random variable

(ii) Continuous random variable

(i) Discrete random variable

If X is the r.v which can take a finite number or countably infinite number of values, X is called a discrete r.v.

For example,

1) The mark obtained by a student in an examination, the possible values are 0, 50, 85, 90.
2) Number of students absented in a particular period.
3) Number of success in n trials.
4) Number of accident in a day in a particular place.
5) Number of telephone calls per unit time.

Probability function or probability mass function

If X is a discrete r.v, which can take the values x_1, x_2, …x_n such that $P(X=x_i)=p_i$then p_i is called the probability function or probability mass function (pmf), provided the following conditions are satisfied:

i) $p_i \geq 0$ for all i.

ii) $\sum_{i=1}^{n} p_i = 1$.

Probability distribution

The collection of pairs $\{x_i,\ p_i\}$ is called probability distribution of random variableX, which is displayed as follows:

x_i	p_i
x_1	p_1
x_2	p_2
x_3	p_3
.	.
.	.
.	.

(ii) Continuous random variable

If X can take all the values (that is infinite number of values) in an interval, then X is called a continuous r.v.

For example,

1. Age, height, weight etc.,
2. The density of milk taken for testing at a farm.
3. The operation time between two failures of a computers.

Probability density function

If X is a continuous r.v such that $P_r\left[a \le x \le b\right]$(or) $P\left\{x - \frac{1}{2}dx \le X \le x + \frac{1}{2}dx\right\} = f(x)dx$,

then f(x) is called probability density function of x provided f(x) satisfying the following conditions:

(i) $f(x) \ge 0 \forall x \in R$.

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$.

Cumulative distribution function

If X is a r.v which is either discrete or continuous, then P(X≤x) is cumulative distribution function (cdf) or distribution function (df) of X and it is denoted by F(x).

$\therefore F(x) = P[X \le x]$.

If X is discrete, then $P\left[X \le x\right] = \sum_{X \le x} p(x)$

If X is continuous, then $P\left[X \le x\right] = \int_{-\infty}^{\infty} f(x)dx$.

Note that, $F(-\infty) = \int_{-\infty}^{\infty} f(x)dx = 0$ (there is no chance for single value in a continuity interval)

Similarly, $F(\infty) = \int_{-\infty}^{\infty} f(x)dx = 1$. (By the total probability)

Note:

If X is discrete r.v, then expectation of a r.v X is defined as E(x) = ∑x P(x). If X is a continuous r.v., then $E(x)=\int_{-\infty}^{\infty} xf(x)dx$.

Problem 1:

If the r.v X takes values 1, 2, 3 and 4 such that 2P(x=1)=3P(x=2)=P(x=3)=5P(x=4). Find the probability distribution and cdf of x.

Solution:

Given X is a discrete r.v(i.e., the values are X= x = 1, 2, 3, 4).

Let 2P(x=1)=3P(x=2)=P(x=3)=5P(x=4) = k

$$\Rightarrow\ 2P(x=1)=k \Rightarrow P(x=1)=\frac{k}{2}$$

$$\Rightarrow\ 3P(x=2)=k \Rightarrow P(x=2)=\frac{k}{3}$$

$$\Rightarrow\ P(x=3)=k \Rightarrow P(x=4)=\frac{k}{5}.$$

Since, the total probability is 1,

$$\frac{k}{2}+\frac{k}{3}+k+\frac{k}{5}=1$$

Simplifies the above equation we get, $k=\frac{30}{61}$

$$\therefore P(x=1)=\frac{k}{2}=\frac{30}{61\times 2}=\frac{15}{61}$$

$$P(x=2)=\frac{k}{3}=\frac{30}{61\times 3}=\frac{10}{61}$$

$$P(x=3)=k=\frac{30}{61}$$

$$P(x=4)=\frac{k}{5}=\frac{30}{61\times 5}=\frac{6}{61}$$

So the probability distribution is

X = x	p(x)
1	15/61
2	10/61
3	30/61
4	6/61

which is the required probability distribution.

To find cdf:-

when $x<1$, $F(x) = 0$

when $x \leq 1$, $F(x)=15/61$

when $x \leq 2$, $F(x)=10/61+15/61=25/61$

when $x \leq 3$, $F(x)=30/61+10/61+15/61=55/61$

when $x \leq 4$, $F(x)=61/61+61/61=1$.

Problem 2:

A r.v X has the following probability distribution:

X=x	-2	-1	0	1	2	3
p(x)	0.1	k	0.2	2k	0.3	3k

(a) Find k (b) Evaluate $P(X<2)$ and $P(-2<x<2)$ (c) Find the cdf of X and (d) Find the mean of X.

Solution:

(a) Since $\sum P(x) = 1$

$$0.6 + 6k = 1$$

$$\therefore k = \frac{1}{15}$$

$\therefore$ The probability distribution is

X=x	-2	-1	0	1	2	3
p(x)	1/10	1/15	2/10	2/15	3/10	3/15

$P(x<2) = P[x = 1, x = 0, x = -1, x = -2]$

= P(x=1) + P(x=0) + P(X=-1) + P(x=-2)

$$= \frac{2}{15} + \frac{2}{10} + \frac{1}{15} + \frac{1}{10} = \frac{1}{2}$$

P[-2<X<2]=P[X=1, X=0, X=-1]

=P[X=-1].P[X=0].P[X=1]

=1/15×2/10×2/15

=2/15.

Find the cdf of x:-

when x < -2 , f(x) = 0

x	p(x)	F(x)=P[X≤x]
-2	1/10	1/10
-1	1/15	1/6
0	2/10	11/30
1	2/15	1/2
2	3/10	4/5
3	3/15	1

When x ≥3, F(x) = 1.

(b) To find the mean of x:-

E(X) = ∑x P(x)

$$= -2\times\frac{1}{10} - 1\times\frac{1}{15} + 0 + 1\times\frac{2}{15} + 2\times\frac{3}{10} + 3\times\frac{3}{15}$$

$$E(x) = \frac{16}{15}.$$

Problem 3:

A r.vX has the following probability distribution:

x	0	1	2	3	4	5	6	7
p(x)	0	k	2k	2k	3k	K^2	$2K^2$	$7K^2+k$

Find (i) value of k (ii) Pr{1.5 < x < 4.5 | x > 2} (iii) The smallest value of λ for which

$Pr\{ x\le\lambda\} > \frac{1}{2}$.

Solution:

(i) To find k:

$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k - 1 = 0$

$k = \frac{1}{10}$ or $k = -1$

$\therefore k = \frac{1}{10}$ (since $0 \le P(x) \le 1$)

$\therefore$ The probability distribution is

x	0	1	2	3	4	5	6	7
p(x)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

ii) $Pr\{1.5 < x < 4.5 \mid x > 2\}$

We know that, $P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$

$$= \frac{P(1.5 < x < 4.5) \cap x > 2}{P(x > 2)}$$

$$= \frac{\frac{2}{10} + \frac{3}{10}}{\frac{2}{10} + \frac{3}{10} + \frac{1}{100} + \frac{2}{100} + \frac{17}{100}}$$

$$= \frac{5}{7}.$$

$\therefore Pr\{1.5 < x , 4.5 / x > 2\} = \frac{5}{7}.$

(iii) $Pr\{ x \le \lambda\} > \frac{1}{2}$

by trail,

put$\lambda = 0, 1, 2, . . .$

$P(x \le 0) = 0$

$P(x \le 1) = \frac{1}{10} = 0.1$

$$P(x \le 2) = \frac{3}{10} = 0.3$$

$$P(x \le 3) = \frac{5}{10} = \frac{1}{2}$$

$$P(x \le 4) = \frac{8}{10} = 0.8$$

The smallest value of λ is 4.

Problem 4:

If $f(x) = \begin{cases} x.e^{\frac{-x^2}{2}} & ; x \ge 0 \\ 0 & ; x < 0 \end{cases}$

(a) Show that f(x) is a pdf of a continuous r.v (b) Find its distribution function of f(x).

Solution:

(a) If f(x) is a pdf. Then

(i) $f(x) \ge 0$ *for all x*

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Obviously $f(x) = x.e^{\frac{x^2}{2}}, x \ge 0$ is a positive

$\therefore P(x) \ge 0 \forall x$

ii)

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

$$= \int_{0}^{\infty} x.e^{\frac{-x^2}{2}} dx$$

Put $t = \frac{x^2}{2} \Rightarrow x^2 = 2t, dt = \frac{1}{2} 2x\,dx \Rightarrow x = \sqrt{2}\,t^{\frac{1}{2}}$

when x=0, t=0, x=∞, t=∞. There is no changes in the limits.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \sqrt{2}\; t^{\frac{1}{2}} e^{-t} \frac{1}{\sqrt{2}\; t^{\frac{1}{2}}} dt$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\therefore$ f(x) is pdf.

b) To find df,

Given $f(x) = \begin{cases} x.e^{\frac{-x^2}{2}} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$

$$\therefore F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{0}^{x} x\,e^{\frac{-x^2}{2}}\,dx$$

Put $t = \frac{x^2}{2}$ dt = x dx

When x = 0, t = 0; x=x, $t = \frac{x^2}{2}$.

$$\therefore F(x) = \int_{0}^{\frac{x^2}{2}} e^{-t} dt$$

$$F(x) = 1 - e\frac{-x^2}{2}$$

$\therefore$ The distribution function is $F(x) = \begin{cases} 1 - e^{\frac{-x^2}{2}} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$.

Problem 5:

If the density function of a continuous r.v X is given by

$$F(x) = \begin{cases} ax & ; 0 \leq x \leq 1 \\ a & ; 1 \leq x \leq 2 \\ 3a - ax & ; 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the value of ‘a’

(ii) Find the cdf of x

(iii) If x_1, x_2 and x_3 are three independent observations of x, what is the probability that exactly one of these three observations is greater than 1.5?

Solution:

i) To find a,

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$,

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 3a - ax\,dx = 1$$

$$\Rightarrow a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + a\left(3x - \frac{x^2}{2}\right)_2^3 = 1$$

After simplification we get, $a = \frac{1}{2}$.

ii) To find the cdf of x

F(x) = P(X≤x)

$$\therefore F(x) = \int_{-\infty}^{x} f(x)\,dx$$

When x <0 , F(x) = 0

When $0 \le x \le 1,\ F(x) = \int_0^x f(x)\,dx$

$$= \int_0^x \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x$$

$$F(x) = \frac{x^2}{4}$$

When $1 \le x \le 2,\ F(x) = \int_0^x f(x)\,dx$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^x a\,dx$$

$$= \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^x \frac{a}{2}\,dx\right]$$

$$F(x) = \frac{x}{2} - \frac{1}{4}$$

When $2 \le x \le 3$, $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_0^x f(x)\,dx$$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \left(\frac{x^2}{4}\right)_0^1 + \left(\frac{x}{2}\right)_1^2 + \left(\frac{3}{2}x - \frac{x^2}{4}\right)_2^x$$

$$\therefore F(x) = \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}$$

When $x \ge 3$, $F(x) = 1$

$$\therefore F(x) = \begin{cases} 0 & ;\text{when } x < 0 \\ \frac{x^4}{4} & ;\text{when } 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{4} & ;\text{when } 1 \le x \le 2 \\ \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4} & ;\text{when } 2 \le x \le 3 \\ 1; & \text{otherwise} \end{cases}$$

iii) The pmf of binomial distribution

$$P(X = x) = nC_x P^x q^{n-x}$$

Here n= 3

Probability of success, $p = \frac{1}{2}$

Probability of failure, $q = \frac{1}{2}$

Pr(exactly greater than 1.5) = P(x>1.5)

$$P(X > 1.5) = 3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{3-1}$$

$$P(X > 1.5) = \frac{3}{8}.$$

1.3 Two Dimensional random variables

Definition

Let S be a sample space which is associated with the random experiment E. Let x = x(s), y = y(s) be the two functions each assigns a real number to each outcomes s $\in$ S. Then the pair (x,y) is called two dimensional r.vs.

There are two types of two dimensional r.vs.

1) Discrete two dimensional r.v
2) Continuous two dimensional r.v.

1) Discrete two dimensional r.v

If the possible values of (x,y) is called two dimensional discrete r.v. When (x,y) is the two dimensional discrete r.v, the possible values of (x,y) may be represented as (x_i, y_j), i = 1, 2, 3, …, m … and j = 1, 2, 3, …, n …

2) Continuous two dimensional rv

If (x, y) can assume all the values in a specified interval or range R with the XY plane, (x, y) is called the two dimensional continuous r.v.

Joint probability mass function of (x, y)

If (x,y) is a two dimensional discrete r.v such that $P(X = x_i, Y = y_i) = P_{ij}$ then P_{ij} is called pmf or probability function of (x, y), provided the following conditions are satisfied

(i) $P_{ij} \geq 0 \forall i, j$

(ii) $\sum_j \sum_i P_{ij} = 1$.

Definition of Joint probability distribution

The collection of triples $\{x_i, y_j, P_{ij}\}$, i = 1, 2 ,. . . , m, . . . j = 1, 2, . . . , n, . . . are called the joint probability distribution of (x, y).

Definition of Joint probability density function

If (x, y) is the two dimensional continuous r.v such that $\Pr\left\{x - \frac{1}{2}dx \le X \le x + \frac{1}{2}dx \ \ and \ y - \frac{1}{2}dy \le Y \le y + \frac{1}{2}dy\right\} = f(x, y)\,dx\,dy$. Then f(x, y) is said to be joint pdfof (x,y) provided the following conditions are satisfied:

(i) $f(x, y) \ge 0 \ \forall \ (x, y) \in R$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$.

Definition of Joint cumulative distribution function

If (x, y) is the two dimensional r.v(discrete or continuous), then $F(x, y) = P(X \le x \ \ and \ \ Y \le y)$ is called cdf of (x,y).

If (x, y) is two dimensional discrete r.v then

$$F(x, y) = P(X \le x \ and \ Y \le y) = \sum_{\substack{y \\ Y \le y}} \sum_{\substack{x \\ X \le x}} P_{ij}$$

If (x, y) is a two dimensional continuous r.v then

$$F(x, y) = P(X \le x \ \ and \ \ Y \le y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\ dx\ dy$$

Note:

(i) $F(x, -\infty) = F(-\infty, y) = 0$

(ii) $F(-\infty, \infty) = 1$.

Marginal probability distribution for the discrete case

Definition

$P(X = x_i, Y = y_i) = \Pr\{(X = x_i \ \ and \ Y = y_1 (\text{or}) (X = x_i \text{ and } Y = y_2)(\text{or})(X = x_i \ and \ y = y_3) or ...\}$

$$= P_{i1} + P_{i2} + \ldots +$$

$$= \sum_j P_{ij}$$

$\therefore P_{i*} = \sum_j P_{ij}$ is called the marginal probability function of x.

i.e., $P(X = x_i) = \sum_j P_{ij} = P_{i*}$

The collection of pairs $\{x_i, P_i*\}$ $i = 1,2,\ldots,m,\ldots$ is called marginal probability distribution of x. Similarly the marginal probability distribution of y is defined by,

$$P(Y = y_i) = \sum_j P_{ij} = p_{*j}$$

The collection of $\{y_j, P_{*j}\}$ j = 1, 2, …, n, … is called the marginal probability distribution of y.

For the continuous case

The marginal density function of X is $f_x(x) = \int_{-\infty}^{\infty} f(x, y)dy$.

The marginal density function of Y is $f_y(y) = \int_{-\infty}^{\infty} f(x, y)dx$.

Definition of conditional probability and conditional probability distribution

For discrete case:

If $P(X = x_i \mid Y = y_i) = \dfrac{P(X = x_i \text{ and } Y = y_i)}{P(Y = y_j)}$

$= \dfrac{P_{ij}}{P_{*j}}$, then $\dfrac{P_{ij}}{P_{*j}}$ is called conditional probability of X given Y.

Then collection of pairs $\left\{x_i, \dfrac{P_{ij}}{P_{*j}}\right\}$ is called conditional probability distribution of X given Y.

Similarly the conditional probability function of Y given X is defined as

$$P(Y = y_j \mid X = x_i) = \frac{P(X = x_i \text{ and } Y = y_i)}{P(X = x_i)}$$

$$= \frac{P_{ij}}{P_{i*}}.$$

The collection of pairs $\left\{y_j, \frac{P_{ij}}{P_{i*}}\right\}$ is called conditional probability distribution of Y given X.

For continuous case:

Conditional density function of X given Y is $F(x \mid y) = \frac{f(x, y)}{f_y(y)}$

Similarly $F(y \mid x) = \frac{f(x, y)}{f_X(x)}$ which is called conditional density function of Y given X.

where f(x, y) is the joint density function

$f_X(x)$ - marginal density function of X

$f_Y(y)$ - marginal density function Y

Definition of Independent r.v or stochastic r.v

If (X, Y) is a two dimensional discrete r.v such that $P(X = x_i \mid Y = y_j) = P(X = x_i)$

i.e., $P(X = x_i \text{ and } Y = y_j) = P(X = x_i) \cdot P(Y = y_j)$

$\Rightarrow P_{ij} = P_{i*} \cdot P_{*j}$

i.e., $P_{ij} = P_{i*} \cdot P_{*j} \; \forall$ i and j.

Then X and Y are called independent random variable.

Similarly, If X and Y are two dimensional continuous r.v such that

$f(x, y) = f_X(x) \cdot f_Y(y)$

Then X and Y are called independent r.v.

Problem 1:

For the bi-variate probability distribution of (X, Y) given below find $P(X \le 1)$, $P(Y \le 3)$, $P(X \le 1, Y \le 3)$, $P(X \le 1 \mid Y \le 3)$, $P(Y \le 3 \mid \le 1)$ and $P(X + Y \le 4)$.

Solution:

x \ y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

$$P(X \le 1) = P(X = 1, X = 0) = \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(0+0+\frac{1}{32}+\frac{2}{32}+\frac{2}{32}+\frac{3}{32}\right)$$

$$P(X \le 1) = \frac{7}{8}.$$

$$P(Y \le 3) = P(Y = 3, Y = 2, Y = 1)$$

$$= \left(\frac{1}{32}+\frac{1}{8}+\frac{1}{64}\right)+\left(0+\frac{1}{16}+\frac{1}{32}\right)+\left(0+\frac{1}{16}+\frac{1}{32}\right)$$

$$P(Y \le 3) = \frac{23}{64}.$$

$$P(X \le 1, Y \le 3) = P(X = 1, X = 0, Y = 3, Y = 2, Y = 1)$$

$$= \left(0+0+\frac{1}{32}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right)$$

$$= \left(0+0+\frac{1}{32}+\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right)$$

$$P(X \le 1, Y \le 3) = \frac{9}{32}.$$

$$P(X \le 1 \mid Y \le 3) = \frac{P(X \le 1 \text{ and } Y \le 3)}{P(Y \le 3)} = \frac{9/32}{23/64}$$

$$P(X \le 1 \mid Y \le 3) = \frac{18}{23}.$$

$$P(Y \le 3 \mid X \le 1) = \frac{P(Y \le 3 \text{ and } X \le 1)}{P(X \le 1)} = \frac{9/32}{7/8}$$

$$P(Y \le 3 \mid X \le 1) = \frac{9}{28}.$$

$$P(X+Y \le 4) = P(X+Y = 1,2,3,4) = 0 + \left(0 + \frac{1}{16}\right) + \left(\frac{1}{32} + \frac{1}{16} + \frac{1}{32}\right) + \left(\frac{2}{32} + \frac{1}{8} + \frac{1}{32}\right)$$

$$P(X+Y \le 4) = \frac{13}{32}.$$

Problem 2:

The joint pmf of (X, Y) is given by P(x,y)=k(2x+3y), x = 0, 1, 2; y= 1, 2, 3. Find all the marginal and conditional probability distributions. Also find the probability distribution of X + Y.

Solution:

P(x,y)=k(2x+3y), x = 0, 1, 2; y= 1, 2, 3

The probability distribution of given function is

x \ y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

Since $\Sigma\Sigma P(x, y) = 1$.

$\Rightarrow$ 3k + 6k + 9k + 5k + 8k + 11k + 7k + 10k + 13k = 1

$$\Rightarrow k = \frac{1}{72}.$$

Therefore, the probability distribution is

x \ y	1	2	3	P_{i*}
0	$\frac{3}{72}$	$\frac{6}{72}$	$\frac{9}{72}$	$\frac{18}{72}$
1	$\frac{5}{72}$	$\frac{8}{72}$	$\frac{11}{72}$	$\frac{24}{72}$
2	$\frac{7}{72}$	$\frac{10}{72}$	$\frac{13}{72}$	$\frac{30}{72}$
P_{*j}	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$	

The marginal probability distribution of X is

X	P_{i*}
0	$\frac{18}{72}$
1	$\frac{24}{72}$
2	$\frac{30}{72}$

The marginal probability distribution of Y is

Y	P_{*j}
1	$\frac{15}{72}$
2	$\frac{24}{72}$
3	$\frac{33}{72}$

The conditional distribution of X given Y = 1 is $\frac{P_{ij}}{P_{*j}} = \frac{P_{i1}}{P_{*1}}$

$$\frac{P_{01}}{P_{*1}} = \frac{3/72}{15/72} = \frac{3}{72} \times \frac{72}{15} = \frac{1}{5}$$

$$\frac{P_{11}}{P_{*1}} = \frac{5/72}{15/72} = \frac{5}{72} \times \frac{72}{15} = \frac{1}{3}$$

$$\frac{P_{21}}{P_{*1}} = \frac{7/72}{15/72} = \frac{7}{72} \times \frac{72}{15} = \frac{7}{15}$$

X	0	1	2
$\frac{P_{ij}}{P_{*j}} = \frac{P_{i1}}{P_{.1}}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{7}{15}$

Similarly we can find the conditional distribution of X given Y = 2

X	$\frac{P_{ij}}{P_{*j}}$
0	$\frac{1}{4}$
1	$\frac{1}{3}$
2	$\frac{5}{12}$

The conditional distribution of X given Y = 2

X	$\frac{P_{ij}}{P_{*j}}$
0	$\frac{9}{33}$
1	$\frac{11}{33}$
2	$\frac{13}{33}$

The conditional distribution of Y given X = 0

Y	$\frac{P_{ij}}{P_{i*}}$
1	$\frac{1}{6}$
2	$\frac{1}{3}$
3	$\frac{1}{2}$

The conditional distribution of Y given X = 1

Y	$\frac{P_{ij}}{P_{i*}}$
1	$\frac{5}{24}$
2	$\frac{1}{3}$
3	$\frac{11}{24}$

The conditional distribution of Y given X = 2

Y	$\frac{P_{ij}}{P_{i*}}$
1	$\frac{7}{30}$
2	$\frac{10}{30}$
3	$\frac{13}{30}$

$$P(X+Y) = P(X+Y = 1,2,3,4,5)$$

$$= \frac{3}{72} + \left(\frac{6}{72} + \frac{5}{72}\right) + \left(\frac{9}{72} + \frac{8}{72} + \frac{7}{72}\right) + \left(\frac{10}{72} + \frac{11}{72}\right) + \frac{13}{72}$$

$$P(X+Y) = 1$$

Problem 3:

The joint pdf of a two dimensional r.v (X, Y) is given by $f(X,Y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le 1$. Compute (i) P(X > 1) (ii) $P\left(Y < \frac{1}{2}\right)$ (iii) $P\left(X > 1 | Y < \frac{1}{2}\right)$ (iv) $P\left(Y < \frac{1}{2} | X > 1\right)$ (v) P(X < Y) (vi) P(X + Y ≤ 1). Also (a) Are X and Y independent? (b) Findthe conditional pdf of X given Y.(c) Find the condition pdf of Y given X.

Solution:

(i) $P(X>1)=\int_0^1\int_1^2 f(x,y)\,dx\,dy$

$$=\int_0^1\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^1\left[y^2.\frac{x^2}{2}+\frac{x^3}{24}\right]_1^2 dy$$

$$=\int\left(y^2\left(\frac{4}{2}\right)+\frac{8}{24}-\frac{y^2}{2}-\frac{1}{24}\right)dy$$

$$=\int_0^1\left(\frac{3y^2}{2}+\frac{7}{24}\right)dy$$

$$=\left[\frac{3}{2}\times\frac{y^3}{3}+\frac{7}{24}\times y\right]_0^1$$

$P(X>1)=\frac{19}{24}$.

(ii) $P(Y<\frac{1}{2})=\int_0^{\frac{1}{2}}\int_0^2 f(x,y)\,dx\,dy$

$$=\int_0^{1/2}\int_0^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^{1/2}\left[y^2.\frac{x^2}{2}+\frac{x^3}{24}\right]_0^2 dy$$

$$=\int_0^{1/2}\left(2y^2+\frac{1}{3}\right)dy$$

$$=\left[\frac{2y^3}{3}+\frac{1}{3}y\right]_0^{1/2}$$

$P\left(Y<\frac{1}{2}\right)=\frac{1}{4}$.

(iii) $P\left(X>1 \mid Y<\frac{1}{2}\right)=\frac{P\left(X>1, Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)}$

$$P\left(X>1 . Y<\frac{1}{2}\right)=\int_0^{\frac{1}{2}}\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^{\frac{1}{2}}\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^{\frac{1}{2}}\left[y^2.\frac{x^2}{2}+\frac{x^3}{24}\right]_1^2 dy$$

$$=\int_0^{\frac{1}{2}}\left(\frac{3y^2}{2}+\frac{7}{24}\right)dy$$

$$=\left[\frac{3y^3}{6}+\frac{7}{24}\right]_0^{\frac{1}{2}}=\frac{5}{24}.$$

But $P\left(Y<\frac{1}{2}\right)=\frac{1}{4}$

$$\therefore P\left(X>1 \mid Y<\frac{1}{2}\right)=\frac{5/24}{1/4}=\frac{5}{6}.$$

(iv) $P\left(Y<\frac{1}{2} \mid X>1\right)=P\left(Y<\frac{1}{2}, X>1\right) \mid P(X>1)$

$$\frac{5/24}{24/19}=\frac{5}{24}\times\frac{24}{19}$$

$$P\left(Y<\frac{1}{2} \mid X>1\right)=\frac{5}{19}.$$

(v) $P(X<Y)=\int_0^1\int_0^y f(x,y)\,dx\,dy$

$$= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8} \right) dx\, dy$$

$$= \int_0^1 \left[y^2 . \frac{x^2}{2} + \frac{x^3}{24} \right]_0^y dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24} \right) dy$$

$$= \left(\frac{y^5}{10} + \frac{y^4}{96} \right)_0^1$$

$$P(X < Y) = \frac{53}{480} .$$

(vi) $P(X + Y \leq 1) = \int\int f(x, y)\, dx\, dy$

$$P(X + Y \leq 1) = \int_0^1 \int_0^{1-y} f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8} \right) dx\, dy$$

$$= \int_0^1 \left[y^2 \frac{x^2}{2} + \frac{x^3}{24} \right]_0^{1-y} dy$$

$$= \int_0^1 \left[\frac{1}{2} \left(\frac{y^3}{3} + \frac{y^5}{5} - \frac{2y^4}{4} \right) + \frac{1}{24} \left(1 - y^3 - 3y + 3y^2 \right) \right] dy$$

$$= \frac{1}{2} \left(+ \frac{1}{5} - \frac{2}{4} \right) + \frac{1}{24} \left(1 - \frac{1}{4} - \frac{3}{2} + \frac{3}{3} \right)$$

$$P(X + Y \leq 1) = \frac{13}{480} .$$

Also,

(a) In order to prove X and Y are independent, we prove

$f(x,y) = f_x(x) . f_y(y)$

$f(X, Y) = xy^2 + \frac{x^2}{8}$; x= 0 to 2; y= 0 to 1

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$= \int_0^1 \left(xy^2 + \frac{x^2}{8} \right) dy$$

$$= \left(x.\frac{y^3}{3} + \frac{x^2}{8} y \right)_0^1$$

$$f_x(x) = \frac{x}{3} + \frac{x^2}{8}, \; 0 \le x \le 2$$

Similarly, $f_y(y) = \int_{-\infty}^{\infty} f(x,y)dx$

$$= \int_0^2 \left(xy^2 + \frac{x^2}{8} \right) dx$$

$$= \left[y^2.\frac{x^2}{2} + \frac{x^3}{24} \right]_0^2$$

$$f_y(y) = 2y^2 + \frac{1}{3}$$

$$f_x(x) \times f_y(y) = \left(\frac{x}{3} + \frac{x^2}{8} \right) \times \left(2y^2 + \frac{1}{3} \right)$$

$$\ne f(x,y)$$

$\therefore$ X and Y are not independent random variables.

(b) Conditional pdf of X given Y is given by

$$f(x \mid y) = \frac{f(x,y)}{f_y(y)}$$

$$= \frac{xy^2 + \frac{x^2}{8}}{2y^2 + \frac{1}{3}}$$

$$f(x \mid y) = \frac{3}{8}.\frac{8xy^2 + x^2}{6y^2 + 1}.$$

(c) Conditional pdf of Y given X is given by

$$f(y \mid x) = \frac{f(x,y)}{f_x(x)}$$

$$= \frac{xy^2 + \frac{x^2}{8}}{2y^2 + \frac{1}{3}}$$

$$= \frac{3}{8} \cdot \frac{8xy^2 + x^2}{\frac{x}{3} + \frac{x^2}{8}}$$

$$f(y \mid x) = 3 \cdot \frac{8xy^2 + x^2}{8x + 3x^2}.$$

1.4 Applications of Jacobian

Let us consider the two-dimensional continuous r.vs X and Y having joint pdf $f(x,y)$. Consider two functions X and Y as U=g(X,Y) and V=h(X,Y) where both the functions are continuous and are differentiable. Then the Jacobian of the transformation is

$$|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

The transformation of U and V is done in such a way that $X = g^{-1}(U, V)$ and $Y = h^{-1}(U, V)$ exist. In such a case J may be positive or may be negative. Then

$$k(u,v) = f(x,y)\,|J|,$$

where f(x, y) is expressed in terms of u and v.

Example 1:

Two-dimensional continuous r.vs X and Y have the joint pdf

$$f(x,y) = \begin{cases} 4xy e^{-(x^2+y^2)}; & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$ Find the pdf of $U = \sqrt{X^2 + Y^2}$.

Solution:

Let $u = \sqrt{x^2 + y^2}$ and $v = x$. $\therefore v \geq 0,\ u \geq 0$. Also $u \geq 0$ and $0 \leq v \leq u$.

The Jacobian of the transformation is

$$\frac{1}{J} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Thus the joint pdf of U and V is

$$g(u,v) = f(x,y)|J| = 4xye^{-(x^2+y^2)}|J|$$

$$= 4x\sqrt{x^2+y^2}\,e^{-(x^2+y^2)}$$

$$= \begin{cases} 4uve^{e^{-u^2}} & ; u \ge 0,\ 0 \le v \le u \\ 0, & \text{otherwise} \end{cases}$$

Hence the density function of $U = \sqrt{X^2+Y^2}$ is

$$h(u) = \int_0^u g(u,v)dv = 4ue^{-u^2}\int_0^v vdv = 2u^3e^{-u^2}, u \ge 0 .$$

$$\therefore h(u) = \begin{cases} 2u^3e^{-u^2}, u \ge 0 \\ 0, \quad \text{otherwise} \end{cases}$$

Example 2:

Let x_1 and x_2 be two observations of a random sample of size 2 from a population having density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, \text{otherwise.} \end{cases}$$

Find the pdf of U = $X_1 + X_2$ and $V = \frac{x_1}{x_1 + x_2}$.

Solution:

Let u = $x_1 + x_2$ and $V = \frac{x_1}{x_1 + x_2}$. Since x > 0, u > 0 and 0 < v < 1.

Here $x_1 = uv$, $x_2 = u(1-v)$.

The Jacobian of the transformation is

$$J=\frac{\partial(x_1,\partial x_2)}{\partial(u,v)}\begin{vmatrix}\frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u}\\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v}\end{vmatrix}=\begin{vmatrix}v & 1-v\\ u & -u\end{vmatrix}=-u$$

We have $f(x_1,x_2)=e^{-(x_1+x_2)}$.

$$\therefore g(u,v)=f(x_1,x_2)|J|=ue^{-u}.$$

Now,

$$g(u)=\int_0^1 g(u,v)dv=ue^{-u}\int_0^1 dv=ue^{-u}, u>0$$

$$\text{and} \quad h(v)=\int_0^\infty ue^{-u}du=1,\ 0<v<1.$$

1.5 Mathematical Expectation

Definition

The mathematical expression for computing the expected value of a r.v X with the pmf / pdf is called the mathematical expectation, which is given below:

$E(X)=\sum x\,P(x)$ for X is discrete r.v.

$E(X)=\int_{-\infty}^{\infty} x\, f(x)\, dx$ for X is continuous r.v.

Properties of Mathematical Expectation

Additive Property

Statement

If X and Y are two r.vs, then E(X+Y)=E(X)+E(Y), provided all the expectation exists.

Proof

Let X and Y be the two continuous r.vs with joint probability function $f_{x,y}(x,y)$ and marginal probability function $f_X(x)$ *and* $f_Y(y)$ respectively.

By the definition of mathematical expectation,

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \qquad (1)$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy \qquad (2)$$

Therefore, $E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{X,Y}(x,y)\, dxdy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\, dxdy$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\right] dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx + \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$E(X+Y) = E(X) + E(Y)$$

Generalization of Additive Property

The mathematical expectation of the sum of n r.vs is equal to the sum of their expectation, provided all the expectation exists.

(i.e.) $E[X_1 + X_2 + ... + X_n] = E[X_1] + E[X_2] + ... + E[X_n]$

$$\Rightarrow E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

Multiplicative Property

Statement

If X and Y are independent r.vs, then E(XY)=E(X) E(Y).

Proof

By the definition of mathematical expectation,

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

$$\text{Therefore } E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x, y)\, dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\, dxdy \text{ (Since X and Y are independent r.vs)}$$

$$= \left(\int_{-\infty}^{\infty} x\, f_X(x)\, dx\right)\left(\int_{-\infty}^{\infty} y\, f_Y(y)\, dy\right)$$

$$E(XY) = E(X)\, E(Y)$$

Generalization of Multiplicative Property

The mathematical expectation of the product of n r.vs is equal to the product of their expectation provided all the expectation exists.

(i.e.) $E[X_1 X_2 \dots X_n] = E[X_1]E[X_2]\dots E[X_n]$

$$\Rightarrow E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$$

Note: 1

If X is the r.v and a and b are the constants then $E[aX+b]=aE(X)+b$.

Proof :

By definition of mathematical expectation,

$$E(X)=\int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E[aX+b]=\int_{-\infty}^{\infty}(ax+b)f(x)\,dx$$

$$=a\int_{-\infty}^{\infty} x\, f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$=a\,E(X)+b(1)$$

$$E[aX+b]=aE(X)+b$$

Note: 2

Expectation of Linear Combination of r.vs.

Let $X_1, X_2, ..., X_n$ be a n r.v and $a_1, a_2, ..., a_n$ are any n constants then $E\left[\sum_{i=1}^{n} a_i X_i\right]=\sum_{i=1}^{n} a_i\, E(X_i)$.

Proof

$$E\left[\sum_{i=1}^{n} a_i\, x_i\right]=E[a_1x_1+a_2x_2+...+a_nx_n]$$

$$=a_1\,E(X_1)+a_2E(X_2)+...+a_nE(X_n)$$

$$=\sum_{i=1}^{n} a_iE(X_i)$$

Problem 1:

Find the Expected value of a Binomial variate.

Solution:

Given X is a Binomial variate, then its pmf given by

$$P(x) = nC_x P^x q^{n-x}, x = 0,1,2,...n$$

$$E(x) = \sum_{x=0}^{n} x\, P(x)$$

$$= \sum_{x=0}^{n} x\, nC_x p^x q^{n-x} x \qquad (\because x = 0,1,2,...n)$$

$$= np \sum_{x=1}^{n} n-1C_{x-1} p^{x-1} q^{n-x}$$

$$= np(q+p)^{n-1}$$

$$= np$$

E(x) = np. Which is the expected value of Binomial Distribution.

Problem 2:

Check whether a continuous r.v with pdf $f(x) = \frac{1}{\pi}.\frac{1}{1+x^2}; -\infty < x < \infty$ is having Expected value of x or not.

Solution:

$$E(x) = \int_{-\infty}^{\infty} |x| f(x) dx$$

$$= \int_{-\infty}^{\infty} |x|.\frac{1}{\pi}.\frac{1}{1+x^2} dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|x|}{1+x^2}dx$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\frac{x}{1+x^2}dx \quad (\therefore\ f(x) \text{ is an even function of } x)$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{2x}{1+x^2}dx = \frac{1}{\pi}\left[\log(1+x^2)\right]_0^{\infty} \to \infty$$

Since this integral does not converge to a finite,

Therefore E(x) does not exist.

Problem 3:

Let X be a r.v with the following probability distribution

x	-3	6	9
p(x)	1/6	1/2	1/3

Find E(x), E(x^2) and using the laws of Expectation, evaluate $E[2x+1]^2$.

Solution:

$$E(x) = \sum_{x=0}^{n} x\,P(x)$$

$$= -3\times\frac{1}{6} + 6\times\frac{1}{2} + 9\times\frac{1}{3}$$

$$E(x) = \frac{11}{2}$$

$$E(x^2) = \sum_{x=0}^{n} x^2\,P(x)$$

$$= 9\times\frac{1}{6} + 36\times\frac{1}{2} + 81\times\frac{1}{3} = \frac{93}{2}$$

$$\therefore E[2x+1]^2 = E\{4x^2+1+4x\}$$

$$= 4E(x^2)+1+4E(x)$$

$$= 4\times\frac{93}{2}+1+4\times\frac{11}{2} = 209$$

$$E[2x+1]^2 = 209$$

Problem 4:

a) Find the expectation of the number on a die when thrown

b) Two unbiased dice are thrown. Find the expected values of the sum of numbers of points on them?

Solution:

a) Let X be the r.v respectively the number on a die when thrown.

$\therefore$ x = {1, 2, 3, 4, 5, 6}

$\therefore$ X can take any one of the values 1, 2, 3, 4, 5, 6 each with equal probability $\frac{1}{6}$.

Hence, E(x) = $\frac{1}{6}\times1+\frac{1}{6}\times2+\frac{1}{6}\times3+\frac{1}{6}\times4+\frac{1}{6}\times5+\frac{1}{6}\times6=\frac{7}{2}$

It means the average toss of a long period one will get $\frac{7}{2}$.

b) The probability function of X for the sum of numbers obtained on two dice is

Values	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(x) = \sum_{x=0}^{n} x\,P(x)$$

$$= 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} + 7\times\frac{6}{36} + 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}$$

$$= \frac{1}{36}[2+6+12+20+30+42+40+36+30+22+12]$$

$$= \frac{1}{36}\times 252$$

=7

E(x)=7

Problem 5:

In four tosses of a coin, let x be the number of heads. Tabulate the 16 possible outcomes with the corresponding values of X. By simple counting, derive the distribution of X and hence calculate the expected value of X

Solution:

The sample space for tossing of a coin four times is

S = {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTTH, THTH, HTHT, THHT, TTHH, HTTT, THTT, TTTH, TTHT, TTTT}

Outcomes	HHHH	HHHT	HHTH	HTHH	THHH	HHTT	HTTH	THTH	HTHT	THHT	TTHH	HTTT	THTT	THTT	TTTH	TTHT	TTTT
No. of Heads	4	3	3	3	3	2	2	2	2	2	2	2	2	1	1	1	0

$$\therefore p(x=0) = \frac{1}{16},\ p(x=1) = \frac{4}{16} = \frac{1}{4},\ p(x=2) = \frac{6}{16} = \frac{3}{8},\ p(x=3) = \frac{4}{16} = \frac{1}{4}\ p(x=4) = \frac{1}{16}$$

$\therefore$ The Probability distribution of X is given by

x	0	1	2	3	4
p(x)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$E(x) = \sum_{x=0}^{n} x\,P(x)$$

$$= 0\times\frac{1}{16} + 1\times\frac{1}{4} + 2\times\frac{3}{8} + 3\times\frac{1}{4} + 4\times\frac{1}{16}$$

$$= \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4}$$

E(x) = 2.

Exercise:

1. Define random variable with examples.
2. Define distribution function and state its properties.
3. A discrete r.v X has the following probability distribution:

X	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

   i) Find a ii) Find $P(x<3)$ iii) Find the variance and mean of x

   iv) Find df of x.

4. A continuous r.v X that can assume any value between x = 2 and x = 5 has the density function given by f(x) = k (1 + x). Find $P(x<4)$.
5. A continuous r.vhas a pdf $f(x) = kx^2 e^{-x}$; $x > 0$.Find k, mean and variance.
6. The df of r.v X is given by $F(X) = 1-(1+x)e^{-x},\ \ x \ge 0.$ Find density function, mean and variance of x.
7. Find cdf for $f(x) = \begin{cases} x e^{-x^2/2} & ; \quad x > 0 \\ 0 & ; \quad x < 0 \end{cases}$.
8. Define Jacobian of transformation.
9. Define marginal and conditional distributions.
10. State and prove additive property of expectation of two random variables.
11. State and prove multiplicative property of expectation of two random variables.

Unit – II

Discrete Distributions

2.1 Introduction

In this Unit, we shall study of the probability distributions that are used most prominently in statistical theory and application. We shall also study their parameter that is the quantities that are constants for particular distributions but that can take on different values for different members of families of distributions of the same kind. We shall introduce number of discrete probability distributions that have been successfully applied in a wide variety of decision situations. The purpose of this Unit is to show the types of situations in which these distributions can be applied.

It may be mentioned that a theoretical probability distribution gives us a law according to which different values of the random variable are distributed with specified probabilities according to some definite law which can be expressed mathematically. It is possible to formulate such laws either on the basis of given conditions (a prior consideration) or on the basic of the results (a posterior inference) of an experiment.

This unit is devoted to the study of univariate discrete distributions like Bernoulli, Binomial, Poisson, Geometric, Negative Binomial, Hyper geometric and Discrete Uniform distributions. We have already defined in the previous Unit about distribution function, mathematical expectation, moment generating function, characteristic function and moments.

2.2 One Point Distribution or Degenerate Distribution

A degenerate distribution or one point distribution is the probability distribution of a r.v which only takes a single value.

Examples include a two-headed coin (biased coin) and rolling a die whose sides all show the same number. This distribution satisfies the definition of "random variable" even though it does not appear random in the everyday sense of the word; hence it is considered degenerate.

The simplest distribution is that of an r.v X degenerate at point k, that is, P{X= k} = 1 and = 0 elsewhere. If we define

$$\varepsilon(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

the distribution function of the r.v X is $\varepsilon(x-k)$. Clearly, $E(X^n)$, n =1,2, …, and $M_x(t)=e^{tk}$. In particular var (X) = 0. This property characterizes a degenerate r.v. The degenerate r.v plays an important role in the study of limit theorems.

2.3 Bernoulli Distribution

Definition

A random variable X which takes two values 0 and 1 with probabilities q and p respectively i.e. P(X=0)=p; P(X=1)=q is called a Bernoulli variate and it is said have a Bernoulli distribution. A random variable X is defined to have a Bernoulli distribution if the discrete density function (or) pmf of X is given by $P(X = x) = \begin{Bmatrix} p^x(1-p)^{1-x} & for\ x = 0\ or\ 1 \\ 0 & otherwise \end{Bmatrix}$ where the parameter p satisfies 0≤p≤1 and q=1-p.

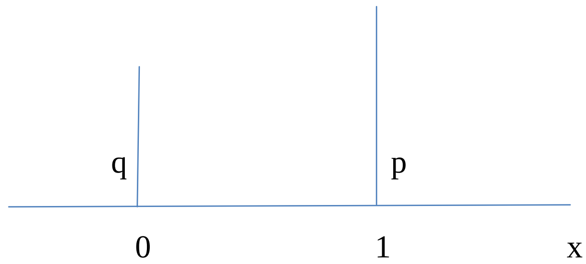

Figure: Bernoulli Density

Definition of moment generating function (MGF)

A moment generating function of r.v X (discrete / continuous) is defined as $M_x(t) = E(e^{tx})$.

For, discrete r.v, $M_x(t) = E(e^{tx}) = \Sigma e^{tx} p(x)$

For, continuous r.v, $M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

Properties of MGF:

(i) The coefficient of $\frac{t^r}{r!}$ in $M_x(t)$ is μ_r'.

By the definition of MGF,

$M_x(t) = E(e^{tx})$

$$= E\left(1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots + \frac{(tx)^r}{r!} + \ldots\right) \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)$$

$$= 1 + \frac{t}{1!} E(x) + \frac{t^2}{2!} E(x^2) + \ldots + \frac{t^r}{r!} E(x^r) + \ldots$$

$$M_x(t) = 1 + \frac{t^1}{1!} \mu_1' + \frac{t^2}{2!} \mu_2' + \ldots + + \frac{t^r}{r!} \mu_r'$$

Therefore, the coefficient of $\frac{t^1}{1!} = \mu_1{}' = Mean$

The coefficient of $\frac{t^2}{2!} = \mu_2{}'$ =Mean square value.

$\therefore\ \mathrm{var(x)} = \mu_2 = \mu_2' - (\mu_1')^2$

(ii) $\dfrac{d^r}{dt^r} M_x(t)\Big|_{t=0} = \mu_r'$

i.e., r^{th} derivative of MGF at t=0 gives μ_r'

MGF and hence Mean and Variance

If X has a Bernoulli distribution, then $E(X) = p,\ V(X) = pq\ and\ M_X(t) = pe^t + q$

Proof:

By the definition of mathematical expectation,

$$E(X) = \sum x P(x)$$

$$= \sum_{x=0}^{1} x\, p^x q^{1-x}$$

$$= 0 + pq^0$$

$$E(X) = p.$$

$\therefore$ Mean= E(x)=p.

$$V(X) = E(X^2) - [E(X)]^2$$

$$\therefore E(X^2) = \sum x^2 P(x)$$

$$= \sum_{x=0}^{1} x^2\ p^x q^{1-x}$$

$$= 0 + p$$

$$E(X^2) = p$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= p - p^2$$

$$= p(1-p)$$

$$V(X) = pq.$$

$\therefore$ Variance=pq.

By the definition of MGF,

$$M_X(t) = E[e^{tx}]$$

$$= \sum e^{tx} P(x)$$

$$= \sum_{x=0}^{1} e^{tx} p^x q^{1-x}$$

$$M_X(t) = q + pe^t.$$

Problem 1:

A random experiment whose outcomes have been classified into two categories called “success” and “failure” represented by the letters ‘s’ and ‘f’ respectively is called a Bernoulli trail. If a random variable X is defined as 1 if a Bernoulli trail results in success and 0 if the same Bernoulli trail results in failure, then X has a Bernoulli distribution with parameter p=Probability of success.

Problem 2:

For a given arbitrary probability space(**Ω**,**A**,P)and for Aε**A**, define the r.v X to be the indicator function of A; that is, $X(\omega) = I_A(\omega)$; then X has a Bernoulli distribution with parameter p=P[X=1]=P[A].

2.4 Binomial Distribution

Definition

A r.v X which takes two values 0 and 1 with probabilities q and p respectively. i.e., P(X=1)p; P(X=0)=q is called a Bernoulli variate and its said have a Bernoulli distribution.

If the experiment is repeated n-times independently with two possible outcome,then they are called Bernoulli trials.

An experiment consisting of a repeated n number of Bernoulli trails is called Bernoulli experiment.

Binomial Experiment

A binomial distribution can be used under the following condition:

(i) Any trail with two possible outcomes that is any trail result in a success or failure.

(ii) The number of trials n is finite and independent,when n is number of trial.

(iii) a probability of success is the same in each trial. i.e., p is the constant.

Definition

A random variable X is said to have a binomial distribution, if its pmf is given by $P(X = x) = \begin{cases} nC_x P^x q^{n-x}, & x = 0,1,2,...n \\ o & ,\text{otherwise} \end{cases}$ where q = 1- p

It is denoted by B(n, p), where n and p are parameters

Applications of Binomial Distribution

1. The quality control measures and sampling process in industries to classify the items are defective or non-defective.
2. Medical applications as a success or failure of a surgery and cure or non cure of a patient.
3. Military application as a hit a target or miss a target

Derivation of mean and variance of B (n, p):

By the definition of mathematical expectation,

$$E(X) = \sum_{x=0}^{n} x\,P(x) = \sum_{x=0}^{n} x\,nC_x\,p^x q^{n-x}$$

$$= np\sum_{x=1}^{n} n-1C_x\; p^{x-1} q^{n-x}$$

= np (q + p) $^{n-1}$ (by binomial expansion)

= np(1) (q+p=1)

Mean = E(x) = np (1)

Var(x) = $E(x^2) - [E(x)]^2$

$$E(x^2) = \sum_{x=0}^{n} x^2 P(x)$$

$$= \sum_{x=0}^{n} [x(x-1) + x]p(x)$$

$$= \sum_{x=0}^{n} x(x-1)p(x) + \sum_{n=0}^{n} xp(x)$$

$$= \sum_{x=0}^{n} x(x-1).nC_x p^x q^{n-x} + np \text{ (From (1))}$$

$$= \sum_{x=0}^{n} x(x-1).\frac{n(n-1)}{x(x-1)} n-2C_{x-2} p^2.p^{x-2} q^{n-x} + np.$$

$$= n(n-1)p^2 \sum_{x=0}^{n} n-2C_{x-2} p^{x-2} q^{n-x} + np$$

$= n (n – 1) p (q + p)^{n-2} + np$

$= n (n - 1) p^2 + np$

$E(x^2) = np (np + q)$

$Var (x) = E(x^2) – [E(x)]^2$

$= np (np + q) – (np)^2$

$= n^2p^2 + npq - n^2p^2$

$Var (x) = npq$

MGF and hence mean and variance

By the definition of MGF,

$$M_x(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= nC_0 (pe^t)^0 q^n + nC_1 (pe^t)^1 q^{n-1} + ... + nC_n (pe^t)^n q^{n-n}$$

$$= q^n + nC_1 (pe^t) q^{n-1} + ... + (pe^t)^n$$

$$M_x(t) = \left(q + pe^t\right)^n$$

Differentiate with respect to t, we get

$$\frac{d}{dt} M_x(t) = n(q + pe^t)^{n-1}.pe^t$$

Put t = 0, $\frac{d}{dt} M_x(t) = n(q + p)^{n-1}.pe^0$

$$\text{Mean} = np = \mu_1'$$

$$\frac{d}{dt} M_x(t) = n(q + pe^t)^{n-1}.pe^t$$

$$= np\left(q + pe^t\right)^{n-1} e^t$$

$$\frac{d^2}{dt^2} M_x(t) = np\left\{(q + pe^t)^{n-1} e^t + e^t (n-1).\left(q + pe^t\right)^{n-2}.pe^t\right\}$$

$$\frac{d^2}{dt^2} M_x(t)\Big|_{t=0} = np\{1 + (n-1)p\}$$

$$np + n^2p^2 - np^2 = \mu_2'$$

$$\therefore \text{var}(x) = \mu_2' - \left(\mu_1'\right)^2$$

$$= np + n^2p^2 - np^2 - (np)^2$$

Var (x) = npq

Definition of Moments

Moments about origin μ_r' is defined as the expectations of the powers of the r.v X. That is $\mu_r' = E(x^r)$. Similarly, the central momentsabout mean is defined as $\mu_r = E(x-\mu)^r$.

Recurrence relation for the central moments of a B(n, p)

By the definition of k^{th} order central moment μ_k is given by

$$\mu_k = E(x-\mu)^k = E(x-np)^k$$

$$= \sum_{x=0}^{n}(x-np)^k nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n}(x-np)^k nC_x p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (x-np)^k p^x (1-p)^{n-x}$$

Differentiate with respect to p, we get

$$\frac{d}{dp}\mu_k = \sum_{x=0}^{n} nC_x \left\{(x-np)^k (p^x (n-x)(1-p)^{n-x-1}(-1) + (1-p)^{n-x}.(xp^{x-1}) + p^x (1-p)^{n-x}.k(x-np)^{k-1}(-n)\right\}$$

After simplification, we get,

$$\frac{d\mu_k}{dp} = -nk\mu_{k-1} + \frac{1}{pq}\mu_{k+1}$$

$$\mu_{k+1} = pq\left[\frac{d\mu_k}{dp} + nk\mu_{k-1}\right] \ldots\ldots (1)$$

Central moments of B(n, p)

Using the above recurrence relation we may compute the moments of higher order, provided the moments of lower order, that is $\mu_0 = 1$ *and* $\mu_1 = 0$.

$$\therefore \mu_{k+1} = pq\left[\frac{d\mu_k}{dp} + nk\mu_{k-1}\right]$$

Put k = 1,

$$\mu_2 = pq\left[\frac{d}{dp}\mu_1 + n\mu_0\right]$$

$$= pq[0 + n]$$

= npq, which is variance of X

$$\therefore \mu_2 = npq$$

Put k = 2,

$$\mu_3 = pq\left[\frac{d}{dp}\mu_2 + 2n\mu_1\right]$$

$$= pq\left[\frac{d}{dp}(npq) + 0\right]$$

= npq(1-2p)

Put k = 3,

$$\mu_4 = pq\left[\frac{d}{dp}\mu_3 + 3n\mu_2\right]$$

$$= pq\left\{\frac{d}{dp}[npq(1-2p)] + 3n(npq)\right\}$$

$$= pq\left\{n\frac{d}{dp}p(1-p)(1-2p) + 3n^2pq\right\}$$

$$= npq\{1 + 3pq(n-2)\}$$

These are the first four binomial central moments.

The first four raw moments (or) moment about origin of B(n, P)

By the definition of moments about origin $\mu_r' = E(x^r)$

To find the first four raw moments:

Put r = 1

$\mu_1' = E(x^1)$

$$= \sum_{x=0}^{n} x p(x)$$

$$= \sum_{x=0}^{n} x \, nC_x p^x q^{n-x}$$

$$= np \sum_{x=0}^{n} n-1C_x p^{x-1} q^{n-x}$$

$$= np\ (q+p)^{n-1}$$

$\mu_1' = np$

$\mu_2' = E(x^2)$

$$= \sum_{x=0}^{n} x^2 p(x)$$

$$= \sum_{x=0}^{n} x(x-1)p(x) + \sum_{x=o}^{n} x\ p(x)$$

$$= n(n-1)p^2 \sum_{x=2}^{n} n-2C_{x-2} p^{x-2} q^{n-x} + np$$

$$= n\ (n-1)P^2\ (q+p)^{n-2} + np$$

$\mu_2' = np(np+q)$

$\mu_3' = E(x^3)$

$$= \sum_{x=0}^{n} x^3 p(x)$$

$$= \sum_{x=0}^{n} [x(x-1)(x-2) + 3x(x-1) + x] nC_x p^x q^{n-x}$$

$$= n(n-1)(n-2)p^3 \sum_{x=0}^{n} n-3C_{x-3} p^{x-3} q^{n-x} + 3n(n-1)p^2 \sum_{x=0}^{n} n-2C_{x-2} p^{x-2} q^{n-x} + np$$

$$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$\mu_4' = E(x^4)$$

$$= \sum_{x=0}^{n} x^4 p(x)$$

$$= \sum_{x=0}^{n} [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x] \; nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3) nC_x p^x q^{n-x} + 6\sum_{x=0}^{n} x(x-1)(x-2) nC_x p^x q^{n-x}$$

$$+ 7\sum_{x=0}^{n} x(x-1) nC_x p^x q^{n-x} + \sum_{x=0}^{n} x \; nC_x p^x q^{n-x}$$

$$= n(n-1)(n-2)(n-3)p^4(p+q)^{n-4} + 6n(n-1)(n-2)p^3(p+q)^{n-3} + 7n(n-1)p^2(p+q)^{n-2} + np$$

$$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$$

Additive property of B(n, p) or Reproductive property

Statement

If $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$, then $X+Y \sim B(n_1+n_2, p)$ where X and Y are independent.

Proof

We know that, the MGF of $B(n, p) = (q+pe^t)^n$.

$\therefore$ The MGF of X ~B(n_1, p) $= (q + pe^t)^{n_1}$.

Also the MGF of Y~B(n_2, P) $= (q + pe^t)^{n_2}$.

We know that, If X and Y are independent r.vs, then

$$M_{X+Y}(t) = M_x(t) \cdot M_x(t)$$

$$= (q + pe^t)^{n_1} . (q + pe^t)^{n_2}$$

$$= (q + pe^t)^{n_1 + n_2}$$

$\therefore M_{X+Y}(t) = (q + pe^t)^{n_1 + n_2}$

Which is the MGF of B(n_1+n_2, p)

$\therefore$ (X+Y) ~ Binomial distribution

Note

If X_1, X_2,..., X_k are independent binomial variates with parameters (n_1,p), (n_2,p),…, (n_k,p) respectively,then $X_1+X_2+\ldots+X_k$ is also a binomial variate with parameter ($n_1+n_2+\ldots+n_k$, p).

Mode of Binomial distribution

Definition

The value of x at which p(x) obtains maximum is called mode of the distribution.

Let X be a binomial r.v. Then $P(X=x)=p(x)=nC_x p^x q^{n-x}$; x = 0, 1, 2,…n

Themode of the binomial distribution is defined by m_0 and it is given by

$$p(m_0-1) \leq p(m_0) \geq p(m_0+1)$$

Consider,

$$p(m_0-1) \leq p(m_0)$$

$$nC_{m_0-1}p^{m_0-1}.q^{n-(m_0-1)} \le nC_{m_0}p^{m_0}q^{n-m_0}$$

$$\Rightarrow \frac{(n-m_0)!m_0!}{(n-m_0+1)!(m_0-1)!}.\frac{q}{p} \le 1$$

$$\frac{m_0}{n-m_0+1} \le \frac{p}{q}$$

$m_0 \le p(n+1)$ ………..... (1)

Consider,

$P(m_0) \ge p\ (m_0 + 1)$

$$nC_{m_0}p^{m_0}q^{n-m_0} \ge nC_{m_0+1}p^{m_0+1}.q^{n-(m_0+1)}$$

$$\Rightarrow \frac{(n-m_0-1)!(m_0+1)!}{(n-m_0)!(m_0)!} \ge \frac{p}{q}$$

$$\frac{m_0+1}{n-m_0} \ge \frac{p}{q}$$

$m_0 \ge np - q$ ……………………….(2)

from (1) and (2)

$np - q \le m_0 \le p\ (n+1)$

For checking:

when n = 10, p=1/2 , q = ½

$4.5 \le m_0 \le 5.5$.

Characteristic function and Cumulative function or cumulative generating function

The characteristic function is defined

$$\varphi_x(t) = E[e^{itx}]$$

Cumulative generating function is defined by

$$\kappa_x(t) = \log\ M_x(t)$$

Characteristic function of B(n,p)

By the definition of characteristic function,

$$\varphi_x(t) = E[e^{itx}]$$

$$= \sum_{x=0}^{n} e^{itx} p(x)$$

$$= \sum_{x=0}^{n} e^{itx} nC_x p^x q^{n-x}$$

$$\varphi_x(t) = (q + pe^{it})^n$$

2.5 Poisson distribution

- Simen Denis Poisson

Definition

A random variable X is said to follow the Poisson distribution if its probability mass function is given by,

$$p(X = x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,...\infty$$

Here the λ is the parameter and $\lambda > 0$

Poisson distribution as a limiting case of Binomial distribution:

Poisson distribution as a limiting case of Binomial distribution under the following condition:

i) The number of trial n is infinitely large. i.e., $n \to \infty$.

ii) The constant probability of success p in each trail is vary small. i.e., $p \to 0$

iii) np = λ is finite, where λ is a positive real number.

Proof:

In the case of Binomial distribution,the probability of x success is given by,

$$p(X=x)=p(x)=nC_x p^x q^{n-x}$$

$$=\frac{n(n-1)(n-2)...[n-(x-1)]}{x!}p^x q^{n-x}$$

Put np = λ; p = λ/n

$$q=1-\frac{\lambda}{n}$$

$$\Rightarrow p(x)=\frac{n(n-1)(n-2)...[n-(x-1)]}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$=\frac{\lambda^x}{x!}.\frac{n}{n}.\frac{n-1}{n}.\frac{n-2}{n}...\frac{n-(x-1)}{n}.\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$=\frac{\lambda^x}{x!}\left[1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)...\left(1-\frac{x-1}{n}\right)\right].\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

Taking limit $n \to \infty$, we get

$$p(X=x)=p(x)=\frac{e^{-\lambda}\lambda^{x}}{x!}, x=0,1,2,...\infty$$

which is the pmf of Poisson distribution.

$\therefore$ Poisson distribution is the limiting case of binomial distribution.

Aliter

The MGF of B(n ,p) is

$$M_x(t)=\left(q+pe^t\right)^n$$

Put np = λ; p = λ/n

$$q=1-\frac{\lambda}{n}$$

$$\therefore M_x(t)=\left(1-\frac{\lambda}{n}+\frac{\lambda}{n}e^t\right)^n$$

$$=\left(1+\frac{\lambda(e^t-1)}{n}\right)^n$$

Taking limit $n\to\infty$ we get

$$p(X=x)=p(x)=\frac{e^{-\lambda}\lambda^{x}}{x!}, x=0,1,2,...\infty$$

which is the MGF of Poisson distribution.

$\therefore$ Poisson distribution is limiting case of Binomial distribution.

Mean and variance of Poisson distribution

Mean, E(x) = $\sum_{x=0}^{\infty} x\,p(x)$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda\lambda^{x-1}}{x(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$\therefore$ Mean E(x) = λ

Variance (x) $= E(x^2) - [E(x)]^2$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 \, p(x)$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x}}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$E(x^2) = \lambda^2 + \lambda$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\text{Var}(x) = \lambda$$

$\therefore$ Mean = Variance = λ.

MGF and hence mean and variance of Poisson distribution

By the definition of MGF,

$$M_x(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda et} = e^{\lambda\left(e^t - 1\right)}$$

$$M_x(t) = e^{\lambda(e^t - 1)}$$

To find mean and variance

By the property of MGF,

$$M_x{}'(t) = e^{\lambda(e^t - 1)}.\lambda(e^t)$$

$$M_x{}'(t)\big|_{t=0} = e^{\lambda(1-1)}.\lambda(e^0) = \lambda$$

$$M_x{}'(t) = \lambda$$

$$\because M_X{}'(t) = \lambda e^t e^{\lambda\left(e^t - 1\right)}$$

$$M_x{}''(t) = \lambda\left[e^t.e^{\lambda(e^t - 1)}.\lambda e^t + e^{\lambda(e^t - 1)}.e^t\right]$$

$$M_x{}''(t)\big|_{t=0} = \lambda[\lambda + 1] = \lambda^2 + \lambda = \mu_2{}'$$

$\text{Var}(x) = \mu_2 = \mu_2' - \left(\mu_1'\right)^2$

$= \lambda^2 + \lambda - \lambda^2$

$\text{Var}(x) = \lambda$

$\therefore$ Mean = Variance = λ.

Recurrence formula for the central moments of the Poisson distribution:

For Poisson distribution with parameter λ; the recurrence formula is,

$$\mu_{r+1} = \lambda\left[\frac{d\mu_r}{d\lambda} + r.\mu_{r-1}\right]$$

Proof

By definition of r^{th} order central moment is given by

$$\mu_r = E(x-\mu)^r$$

$$= E(x-\lambda)^r \qquad (\because E(x) = \lambda)$$

$$= \sum_{x=0}^{\infty}(x-\lambda)^r.p(x)$$

$$\mu_r = \sum_{x=0}^{\infty}(x-\lambda)^r\,\frac{e^{-\lambda}\lambda^x}{x!}$$

Differentiate with respect to λ, we get,

$$\frac{d}{d\lambda}\mu_r = \sum_{x=0}^{\infty}\frac{1}{x!}\left[(x-\lambda)^r.(e^{-\lambda}x\lambda^{x-1} + \lambda^x e^{-\lambda}(-1)) + (e^{-\lambda}\lambda^x).r(x-\lambda)^{r-1}(-1)\right]$$

$$\Rightarrow \lambda\frac{d\mu_r}{d\lambda} = \mu_{r+1} - \lambda r\ \mu_{r-1}$$

$$\Rightarrow \mu_{r+1} = \lambda \frac{d\mu_r}{d\lambda} + \lambda r \ \mu_{r-1}$$

$$\Rightarrow \mu_{r+1} = \lambda \left[\frac{d\mu_r}{d\lambda} + r\mu_{r-1} \right].$$

The central moments μ_1, μ_2, μ_3 and μ_4:

The recurrence formula for central moments of Poisson distribution is,

$$\mu_{r+1} = \lambda \frac{d\mu_r}{d\lambda} + \lambda r \ \mu_{r-1} \qquad \ldots\ldots\ldots\ldots\ldots(*)$$

Also, we know that, $\mu_0 = 1$

$\mu_1 = 0.$

In order to get μ_2, put r=1 in (*),

$$\therefore \mu_2 = \lambda \frac{d\mu_1}{d\lambda} + \lambda \mu_0$$

$$= \lambda x_0 + \lambda x_1$$

$$\mu_2 = \lambda .$$

In order to get μ_3,Put r = 2 in (*),

$$\therefore \mu_3 = \lambda \frac{d\mu_2}{d\lambda} + 2\lambda \mu_{2-1}$$

$$= \lambda.1 + 2\lambda(0)$$

$$\mu_3 = \lambda$$

In order to get μ_4,Put r = 3 in (*),

$$\therefore \mu_4 = \lambda \frac{d\mu_3}{d\lambda} + 3\lambda\,\mu_2$$

$$= \lambda.1 + 3\lambda.\lambda$$

$$\mu_4 = \lambda + 3\lambda^2$$

$\therefore$ $\mu_1 = 0$, $\mu_2 = \lambda$, $\mu_3 = \lambda$, $\mu_4 = \lambda + 3\lambda^2$ are the first four central moments.

The first four moments about origin:

By the definition of r^{th} order raw moments,

$$\mu_r' = E[x^r]$$

$$\therefore \mu_1' = E(x) = E(x)$$

$$= \sum_{x=0}^{\infty} x\,.\,p(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda\lambda^{x-1}}{x(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$\mu_1' = \lambda$$

Also, $\mu_2' = E(x^2)$

$$\mu_2' = \sum_{x=0}^{\infty} x^2\, p(x)$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x] \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\mu_2{}' = \lambda^2 + \lambda$$

Also, $\mu_3{}' = E(x^3)$

$$\mu_3{}' = \sum_{x=0}^{\infty} x^3\, p(x)$$

$$= \sum_{x=0}^{\infty} x^3 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1)(x-2) + 3x(x-1) + x] \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} 3x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^3 e^{\lambda} + 3e^{-\lambda}\lambda^2 e^{\lambda} + \lambda$$

$$\mu_3{}' = \lambda^3 + 3\lambda^2 + \lambda$$

Also $\mu_4{}' = E(x^4)$

$$\mu_4{}' = \sum_{x=0}^{\infty} x^4\, p(x)$$

$$= \sum_{x=0}^{\infty} x^4 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x] \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)(x-3) \frac{e^{-\lambda}\lambda^4\lambda^{x-4}}{x(x-1)(x-2)(x-3)(x-4)!}$$

$$+ 6\sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-\lambda}\lambda^3\lambda^{x-3}}{x(x-1)(x-2)(x-3)!}$$

$$+ 7\sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^2\lambda^{x-2}}{x(x-1)(x-2)!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda .$$

Additive property:

The sum of independent Poisson variates is also a Poisson variate.

i.e., X_1, X_2, …. X_n are n independent Poisson variates with parameter λ_1, λ_2, …. λ_n. Then $X_1 + X_2 + \ldots + X_n$ is also a Poisson variate with parameter $\lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Proof:

We know that the MGF of Poisson distribution is,

$$M_x(t) = e^{\lambda(e^t - 1)}$$

Also we know that,

$$M_{x_1+x_2+\ldots+x_n}(t) = M_{x_1}(t).M_{x2}(t)\ldots M_{x_n}(t)$$

$$= e^{\lambda_1(e^t-1)} + e^{\lambda_2(e^t-1)} + \ldots\ldots + e^{\lambda_n(e^t-1)}$$

$\therefore M_{X_1+X_2+\ldots+X_n}(t) = e^{(\lambda_1+\lambda_2+\ldots+\lambda_n)(e^t-1)}$ which is the MGF of $X_1 + X_2 + \ldots + X_n$ with parameter $\lambda_1 + \lambda_2 + \ldots + \lambda_n$.

$\therefore$ $X_1 + X_2 + \ldots + X_n$ is also Poisson variate.

Examples of a Poisson distribution (Real life Problems)

1. Number of printing mistakes at each page of a book.
2. The number of road accident reported in a city per day
3. The number of death in a district due to rare disease.
4. The number of defective articles in a pocket of 200
5. The number of cars passing through a time interval t.

Theorem 1

If X and Y are two independent Poisson variates with parameters λ_1, λ_2,then the conditional distribution of $\left(X \middle| X+Y\right)$ is Binomial.

Proof

Given X and Y are independent Poisson variates with parameter λ_1 and λ_2 respectively.

$$\therefore P(X=m) = \frac{e^{-\lambda_1}\lambda_1^{\ m}}{m!} \ ; X = 0, 1, 2, \ldots , m, \ldots$$

$$\therefore P(Y=n) = \frac{e^{-\lambda_2}\lambda_2^{\ n}}{n!} \ ; Y = 0, 1, 2, \ldots , n, \ldots$$

$$\therefore P(X|X+Y) = P(X=m|X+Y=n)$$

$$= \frac{P(X=m, X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=m, Y=n-m)}{P(X+Y=n)}$$

$$= \frac{P(X=m)P(Y=n-m)}{P(X+Y=n)}$$

$\therefore$ X and Y are independent.

$$= \frac{\dfrac{e^{-\lambda_1}\lambda_1^{\ m}}{m!} \cdot \dfrac{e^{-\lambda_2}\lambda_3^{\ n-m}}{(n-m)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$

Multiply and divide by $\left(\dfrac{n!}{\lambda_1+\lambda_2}\right)^m$

$$= \frac{n!}{(n-m)!m!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}$$

$$= nC_m p^m q^{n-m} \text{ where } p = \frac{\lambda_1}{\lambda_1+\lambda_2} \text{ and } q = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

Which is the pmf of binomial distribution.

$\therefore$ If X and Y are two independent Poisson variate, then the condition probability of $X|X+Y$ is Binomial.

Theorem 2

If X is a Poisson variate with parameter λ and conditional distribution of $y \mid x$ follows binomial with parameters n and p, then the distribution of Y follows the Poisson distribution with parameter λp.

Proof

Given X is a Poisson variate with parameter λ.

$$\therefore P(X = x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,...\infty$$

For a Binomial distribution $P(X = x) = p(x) = nC_x p^x q^{n-x}; x = 0, 1, 2, ...n$

Then we prove that,Y~Poisson (λp)

$$\therefore P[Y = m|X = n] = \frac{P(Y = m, X = n)}{P(X = n)}$$

$$\Rightarrow P(X = n, Y = m) = P(Y = m|X = n).P(X = n)$$

$$= nC_m p^m q^{n-m} . \frac{e^{-\lambda}\lambda^n}{n!} \qquad (1)$$

$$\therefore P[Y = m] = \sum_{x=0}^{\infty} P(X = n, Y = m)$$

$$= \sum_{n=m}^{\infty} nC_m p^m q^{n-m} . \frac{e^{-\lambda}\lambda^x}{x!} \text{ (from (1))}$$

$$= \frac{e^{-\lambda} p^m \lambda^m}{m!} \sum_{n=m}^{\infty} \frac{(\lambda q)^{n-m}}{(n-m)!}$$

$$= \frac{e^{-\lambda p} (\lambda p)^m}{m!}$$

which is the pmf of Poisson distribution with parameter is λp.

$\therefore$ If X~Poisson (λ) and $Y|X \sim B(n, p)$, then Y ~ Poisson(λp).

Theorem 3

If X and Y are two independent Poisson variates then X-Y is not a Poisson variate.

Proof

Given,

$$M_x(t) = e^{\lambda_1 (e^t - 1)}$$

$$M_y(t) = e^{\lambda_2(e^t - 1)}$$

$$M_{x-y}(t) = M_x(t).M_{(-y)}(t)$$

$$= M_x(t).M_y(-t)$$

$$= e^{\lambda_1(e^t - 1)}.e^{\lambda_2(e^{-t} - 1)}$$ which is not in the form of $e^{\lambda(e^t - 1)}$.

So difference X-Y is not a Poisson variate.

2.6 Geometric Distribution (x-1 failures preceding the first success)

Definition

A random variable X is said to follow a Geometric distribution if it assumes the non-negative value and its pmf is given by,

$$P(X = x) = \begin{cases} q^{x-1}p & x = 1,2,3,\ldots,\infty \\ 0 & otherwise \end{cases}$$

Here p is the parameter, q=1-p, 0≤p≤1

(or)

$$P(X = x) = \begin{cases} q^{x}p & x = 0,1,2,3,\ldots,\infty \\ 0 & otherwise \end{cases}$$

x failures preceding the first success.

Note

1. $P(X = x) = q^{x-1}p$ denotes the probability that there are x-1 failure preceding the first success.
2. We know that, the total probability is one.

$$\text{That is, } P(X = x) = q^{x-1}p$$

$$\Rightarrow \sum_{x=1}^{\infty} P(X = x) = \sum_{x=1}^{\infty} q^{x-1}p$$

$$= p + pq + pq^2 + \ldots$$

$$= p[1 + q + q^2 + \ldots]$$

$$= p(1-q)^{-1}$$

$$= \frac{p}{1-q} = \frac{p}{p}$$

$$\sum_{x=1}^{\infty} P(X = x) = 1$$

The total probability is 1.

3. Prove that for a Geometric distribution, the variance is always greater than mean.

Solution

For Geometric Distribution,Mean = $\frac{1}{p}$.

Variance = $\frac{q}{p^2}$

$= \frac{q}{p}\frac{1}{p}$

Variance(X) = $\frac{q}{p} \times mean$

Variance(X) > Mean.

Mean and Variance:

By the definition of Expectation,

$$E(X) = \sum_{x=1}^{\infty} x.p(x)$$

$$= \sum_{x=1}^{\infty} x.q^{x-1}p$$

$$= p\sum_{x=1}^{\infty} x.q^{x-1}$$

$$= p[1 + 2q + 3q^2 + ...]$$

$$= p(1-q)^{-2}$$

$$=\frac{p}{p^2}$$

$$E(X)=\frac{1}{p}$$

$$Var(X)=E(x^2)-[E(x)]^2$$

$$=\frac{2q}{p^2}+\frac{1}{p}-\left(\frac{1}{p}\right)^2$$

$$=\frac{2q+p-1}{p^2}$$

$$Var(X)=\frac{2q-q}{p^2}=\frac{q}{p^2}$$

MGF and hence Mean and Variance:

By definition of mgf,

$$M_X(t)=\frac{pe^t}{(1-qe^t)}$$

$$M_X(t)=pe^t(1-qe^t)^{-1}$$

$$\therefore M_X'(t)=p\left\{e^t(-1)(1-qe^t)^{-2}(-qe^t)+(1-qe^t)^{-1}e^t\right\}$$

$$M_X'(t)|_{t=0}=p\left\{\frac{q}{p^2}+\frac{1}{p}\right\}$$

$$=\frac{p+q}{p}$$

$$=\frac{1}{p}\qquad=\mu_1'$$

$$\therefore \text{Mean } \mu_1'=\frac{1}{p}$$

$$M_X'(t)=P\left\{qe^{2t}(1-qe^t)^{-2}+(1-qe^t)^{-1}e^t\right\}$$

$$M_X''(t) = p\{qe^{2t}(-2)(1-qe^t)^{-3}(-qe^t) + (1-qe^t)^{-2} qe^{2t}\, 2) + (1-qe^t)^{-1} e^t + e^t(-1)(1-qe^t)^{-2}(-qe^t)\}$$

$$M_X''(t)|_{t=0} = p\{2q^2(1-q)^{-3} + (1-q)^{-2} 2q + (1-q)^{-1} + q(1-q)^{-2}\}$$

$$= p\left\{\frac{2q^2}{p^3} + \frac{2q}{p^2} + \frac{1}{p} + \frac{q}{p^2}\right\}$$

$$= \frac{2q^2}{p^2} + \frac{2q}{p} + \frac{1}{p}$$

$$= \frac{2q^2 + 2pq + p}{p^2}$$

$$= \frac{2q^2 + 2(1-q)q + 1 - q}{p^2}$$

$$= \frac{1 + q - 1}{p^2}$$

$$M_X''(t)|_{t=0} = \mu_2'$$

$$\text{Var}(X) = \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2$$

$$\text{Var}(X) = \frac{q}{p^2}$$

Memory less Property of Geometric Distribution

Statement

If X has a geometric distribution then for any two positive numbers m and n, $P(X > m+n \mid X > m) = P(X > n)$. We need m+n trials for getting first success, given than m consecutive failures is equal to the unconditional probability of at least n trials to get their success. Here m failure is not in memory.

Proof

$$P(X > m+n \mid X > m) = \frac{P(X > m+n \cap X > m)}{P(X > m)}$$

$$= \frac{P(X > m+n)}{P(X > m)}$$

The pmf of geometric distribution is,

$$P(X = x) = q^{x-1}p; \qquad x = 1,2,3,\ldots,\infty$$

for any k,

$$P(X > k) = \sum_{x>k} p(x)$$

$$= \sum_{x=k+1}^{\infty} q^{x-1} p$$

$$= q^{k+1-1} p + q^{k+2-1} p + \ldots$$

$$= q^{k} p + q^{k+1} p + \ldots.$$

$$= p q^{k} [1 + q + q^{2} + \ldots]$$

$$= p q^{k} (1 - q)^{-1}$$

$$P(X > k) = q^{k}$$

$$P(X > m + n) = q^{m+n}$$

$$P(X > m) = q^{m}$$

$$p(X > n) = q^{n}$$

$$P(X > m + n \mid X > m) = \frac{P(X > m + n)}{P(X > m)}$$

$$= \frac{q^{m+n}}{q^{m}}$$

$$P(X > m + n \mid X > m) = q^{n}.$$

2.7 Negative Binomial Distribution

Let X denote the number of failure preceding the r^{th} success, then P(X=x) denotes the probability that there are x failure preceding the r^{th} success in x+r trial. Clearly the last trial is success with a probability P.

In the preceding (x+r-1) we must have a x failures and r-1 success, in any order, whose probability function is given by,

$$x+r-1_{C_{r-1}} P^{r-1} q^{x}$$

Hence the probability of x failures and r^{th} success is given by,

$$P(X=x)=x+r-1_{C_{r-1}} P^{r} q^{x}$$

Definition

A random variable X is said have a negative binomial distribution if its probability mass function is given by,

$$P(X=x)=x+r-1_{C_{r-1}} P^{r} q^{x}; \qquad x=0,1,2,...$$

Here p and r are parameters.

(or) $P(X=x)=-r_{C_x} P^{r}(-q)^{x}; \quad x=0,1,2,...$

Note

Geometric distribution is a special case of negative binomial distribution when r=1.

MGF and hence Mean and Variance

By definition of mgf,

$$M_X(t)=E[e^{tx}]$$

$$=\sum_{x=0}^{\infty} e^{tx} p(x)$$

$$=\sum_{x=0}^{\infty} e^{tx}\, x+r-1_{C_{r-1}} p^{r} q^{x}$$

$$=p^{r}\sum_{x=0}^{\infty} (e^{t}q)^{x}\, x+r-1_{C_{r-1}}$$

$$=p^{r}\left[1+rqe^{t}+(qe^{t})^{2}\frac{(r+1)r}{2!}+...\right]$$

$$=p^{r}\left[1-qe^{t}\right]^{-r} \quad =\left[\frac{p^{r}}{1-qe^{t}}\right]^{r}$$

$$M_X(t) = \frac{p^r}{(1-qe^t)^r}.$$

To find Mean and Variance

We know that

$$M_X(t) = P^r[1-qe^t]^{-r}$$

$$M_X{}'(t) = P^r(-r)(1-qe^t)^{-r-1}(-qe^t)$$

$$M_X{}'(t)|_{t=0} = P^r(-r)(1-q)^{-(r+1)}(-q)$$

$$= rP^r q(P)^{-(r+1)}$$

$$= \frac{rP^r q}{P^{r+1}} = \mu_1{}'$$

$$\therefore \text{Mean} = \frac{rq}{P}$$

$$M_X{}'(t) = rqP^r(1-qe^t)^{-(r+1)}e^t$$

$$M_X{}''(t) = rqP^r\left\{(1-qe^t)^{-(r+1)}e^t + e^t(-(r+1))(1-qe^t)^{-(r+2)}(-qe^t)\right\}$$

$$M_X{}''(t)|_{t=o} = rqP^r\left\{(1-q)^{-(r+1)}e^0 + e^0(-(r+1))(1-q)^{-(r+2)} - q\right\}$$

$$= rqP^r\left\{P^{-(r+1)} + (r+1)qP^{-(r+2)}\right\}$$

$$= \frac{rqP^r}{P^{r+1}} + \frac{rqP^r(r+1)q}{P^{r+2}}$$

$$= \frac{rq}{P} + \frac{rq^2(r+1)}{P^2}$$

$$= \mu_2{}'$$

$$\mu_2{}' = \frac{rq}{P} + \frac{r(r+1)q^2}{P^2}.$$

$$\therefore Var(X) = \mu_2' - (\mu_1')^2$$

$$= \frac{rq}{P} + \frac{r(r+1)q^2}{P^2} - \left(\frac{rq}{P}\right)^2$$

$$= \frac{rq}{P}\left[1 + \frac{(r+1)q}{P} - \frac{rq}{P}\right]$$

$$= \frac{rq}{P}\left[\frac{P + (r+1)q - rq}{P}\right]$$

$$Var(X) = \frac{rq}{P^2}.$$

Additive Property of Negative Binomial Distribution

Statement

Let X_1 *and* X_2 be the two independent negative binomial variates with same parameter p and different numbers of successes r_1 *and* r_2 then the sum $X_1 + X_2$ is also a negative binomial variate with parameters p and $r_1 + r_2$.

Proof

We know that

The mgf of Negative Binomial Distribution is,

$$M_X(t) = P^r(1 - qe^t)^{-r}$$

$$M_{X_1}(t) = P^{r1}(1 - qe^t)^{-r1}$$

$$M_{X_2}(t) = P^{r2}(1 - qe^t)^{-r2}$$

$$\therefore M_{X_1+X_2}(t) = M_{X_1}(t).\, M_{X_2}(t)$$

$$= P^{r1}(1 - qe^t)^{-r1} \times P^{r2}(1 - qe^t)^{-r2}$$

$$= P^{r1+r2}(1 - qe^t)^{-(r1+r2)}$$

which is the mgf of Negative Binomial Distribution.

2.8 Hypergeometric Distribution

If X represents the number of defectives found, when n items are drawn without replacement from a lot of items containing k defectives and (N-k) non-defectives, clearly

$$P(X = r) = \frac{kC_r (N-k)C_{(n-r)}}{NC_n}; r = 0,1,2,\ldots,\min(n,k)$$

Note:

If n > k, then the maximum value of X is k;If n < k, then the maximum value of X is n,

That is, the maximum value of X is min(n, k), that is, r can take the values 0, 1, 2, …, min(n, k)

Definition:

If X is a discrete r.v that can assume, non-negative values 0, 1, 2,…, such that its probability mass function is given by

$$P(X = r) = \frac{kC_r (N-k)C_{(n-r)}}{NC_n}; r = 0,1,2,\ldots,\min(n,k)$$

Then X is said to follow a hypergeometric distribution with the parameters N, k and n.

Note:

1. In the probability mass function of X, r can be assumed to take the values 0, 1, 2, …, n, which is true when n < k. But when n > k, r can take the values 0, 1, ,2, … , k. In other words,P(X=r)=0, when r = k +1, k+2,…n. This values (namely zero) of the probability is provided by the probability mass function formula itself, since kC_r = 0, for r =k+1, k+2,…,n.Thus in the values of P(X=r) min(n, k) can be replaced by n.

2. Hypergeometric distribution is a legitimate probability distribution, since

$$\sum_{r=0}^{n} P(X = r) = \sum_{r=0}^{n} \frac{kC_r (N-k)C_{(n-r)}}{NC_n}$$

$$= \frac{1}{NC_n} NC_n = 1 \text{ since}$$

$$\sum_{r=0}^{n} kC_r (N-k)C_{n-r} = coefficien\,tof \;\; X^n \; in (1+x)^k (1+x)^{N=k}$$

= coefficient of X^n in (1+X) = NC_n

Mean and Variance of Hypergeometric Distribution

$$E(X) = \sum_r x_r P_r$$

$$= \frac{k}{NC_n} \sum_{r=1}^{n} (k-1)C_{(r-1)} (N-k)C_{(n-r)}$$

$$= \frac{k}{NC_n} \sum_{r'=0}^{n-1} k'C_{r'} (N-1-k')C_{(n-1-r')}$$

(on putting $k' = k-1 \;\; and \;\; r' = r-1$)

$$= \frac{k}{NC_n} .(N-1)C_{n-1}$$

$$= \frac{nk}{N}$$

$$E(X^2) = E\{X(X-1) + X\}$$

$$= E\{X(X-1)\} + \frac{nk}{N}$$

$$= \frac{nk}{N} + \sum_{r=0}^{n} (r(r-1)kC_r .(N-k)C_{(n-r)} / NC_n$$

$$= \frac{nk}{N} + \frac{k(k-1)}{NC_n} \sum_{r'=0}^{n-2} k'C_{r'} .(N-2-k')C_{(n-2-r')}$$

(on putting $k' = k-2 \, and \, r' = r-2$)

$$\text{k} = \frac{nk}{N} + \frac{k(k-1)}{NC_n} .(N-2)C_{n-2}$$

$$= \frac{nk}{N} + \frac{k(k-1)n(n-1)}{N(N-1)} .(N-2)C_{n-2}$$

$$\text{Var(X)} = \text{E}(X^2) - \{\text{E(X)}\}^2$$

$$= \frac{nk}{N} + \frac{k(k-1)n(n-1)}{N(N-1)} + \frac{n^2 k^2}{N^2}$$

$$= \frac{nk}{N^2(N-1)}[N(N-1) + N(k-1)(n-1) - (N-1)nk]$$

$$= \frac{nk(N-k)(N-n)}{N^2(N-1)}$$

Note:

If we denote the proportion of defective items in the lot as p, i.e., $p = \frac{k}{N}$ and q = 1 – p, then E(X) = np and Var(x) = npq $\left[\frac{N-n}{N-1}\right]$

Binomial Distribution as Limiting form of Hypergeometric Distribution

Hypergeometric distribution tends to binomial distribution as $N \to \infty$ and $\frac{k}{N} = p$

Proof:

If X follows a hypergeometric distribution with parameters N, k and n, then

$$P(X = r) = \frac{kC_r\,(N-k)C_{(n-r)}}{NC_n}, r = 0,1,2,...,n$$

$$\frac{k(k-1)...(k-r+1)}{r!} \cdot \frac{(N-k)(N-k-1)...(N-k-n+r+1)}{(n-r)!} \times \frac{n!}{N(N-1)...(N-n+1)}$$

$$= \frac{n!}{r!(n-r)!} \times \frac{\left[\left(\frac{k}{N}\right)\left(\frac{k}{N} - \frac{1}{N}\right)...\left\{\frac{k}{N} - \frac{(r-1)}{N}\right\}\right]\left[\left(1 - \frac{k}{N}\right)\left\{1 - \frac{k+1}{N}\right\}...\left\{1 - \frac{k+n-r-1}{N}\right\}\right]}{1.\left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)...\left\{1 - \frac{(n-1)}{N}\right\}}$$

(by dividing each factor in the numerator and denominator ny N)

Putting $\frac{k}{N} = p$ and proceeding to the limit as $N \to \infty$, we get

$$\lim_{\substack{N \to \infty \\ \frac{k}{N} = p}} p\{X = r\} = nC_r . p^r (1-p)^{n-r}$$

$$= nC_r \, p^r q^{n-r}; r = 0,1,2,..., n$$

Thus the limit of a hypergeometric distribution is a binomial distribution.

Note:

We know that the binomial distribution holds good when we draw samples with replacement (since the probability of getting a defective item has to remain constant), while the hypergeometric distribution olds good when we draw samples without replacement. If the lot zize N is very large, there is not much difference in the proportions of defective items in the lot whether the item drawn is replaced or not. The previous result is simply a mathematical statement of this fact.

2.9 Multinomial Distribution

Definition

Multinomial distribution is the generalization of binomial distribution. Consider k events $E_1, E_2, \ldots, E_k$. The event E_1 occurs X_1 times, E_2 occurs X_2 times and so on, with the corresponding probability $p_1, p_2, \ldots p_k$ respectively.

Let us assume that the probability of getting i^{th} event in x_i times is $p_i^{x_i}$, i = 1, 2, 3, …,k. Then the joint probability function of k events is given by,

$$p(x_1, x_2, ... x_k) = \frac{n!}{x_1! x_2! .... x_k!} . p_1^{x_1} . p_2^{x_2} ... p_k^{x_k}$$

$$= \frac{n!}{\prod_{i=1}^{k} x_i!} \prod_{i=1}^{k} p_i^{x_i}$$

This distribution is called multinomial distribution,where $(p_1+p_2+ \ldots +p_k) = 1$, $n = x_1+x_2+ \ldots + x_k$.

For example, if a fair die is tossed twelve times, the probability of getting 1, 2, 3, 4, 5 and 6 points exactly twice each is given by

$$p(x_1 = 2, x_2 = 2, x_3 = 2, x_4 = 2, x_5 = 2, x_6 = 2) = \frac{12!}{2!\,2!\,2!\,2!\,2!\,2!} \times \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2$$

$$= 0.00344$$

MGF of Multinomial Distribution

To derive MGF, first let us consider a trail which has two outcomes A_1, A_2.

Assume the outcome A_1 occurs x_1 times and A_2 occurs x_2 times then the probability of getting A_1, x_1 times and A_2, x_2 times is given by the function,

$$p(x_1, x_2) = \frac{n!}{x_1!x_2!} p_1^{x_1} p_2^{x_2} \qquad \text{where } p_1 + p_2 = 1 \text{ and } n = x_1 + x_2$$

$$M_{x_1,x_2}(t) = E(e^{t_1x_1 + t_2x_2})$$

$$= \sum_x e^{t_1x_1 + t_2x_2} . p(x_1, x_2)$$

$$= \sum_x e^{t_1x_1 + t_2x_2} \frac{n!}{x_1!x_2!} p_1^{x_1} . p_2^{x_2}$$

$$= \left(p_1 e^{t_1} + p_2 e^{t_2}\right)^n$$

Which is the MGF of $p(x_1, x_2)$. By simply extending this result the mgf for multinomial distribution can be written as,

$$M_{x_1,x_2,\ldots x_k}(t) = \left(p_1 e^{t_1} + p_2 e^{t_2} + \ldots + p_k e^{t_k}\right)^n$$

From this MGF, we can find mean and variance as follows:

$$\therefore M_x(t) = \left(p_1 e^{t_1} + p_2 e^{t_2} + \ldots . + p_k e^{t_k}\right)^n$$

$$\frac{d}{dt_i}M_x(t) = n\left(p_1e^{t_1} + p_2e^{t_2} + ... + p_ke^{t_k}\right)^{n-1} \times p_ie^{t_i}$$

$$\frac{d}{dt_i}M_x(t)\Big|_{t_i=0} = n\left(p_1 + p_2 + ... + p_k\right)^{n-1} \times p_i$$

$$= np_i$$

$$= \mu_1'$$

Mean $= \mu' = np_i$

$$\because \frac{d}{dt_i}M_X(t) = np_i\left[e^{t_i}\left(p_1e^{t_1} + p_2e^{t_2} + ... + p_ke^{t_k}\right)\right]$$

$$\frac{d^2}{dt_i^{\,2}}M_x(t) = np_i\left[e^{t_i}(n-1)\left(p_1e^{t_1} + p_2e^{t_2} + ... + p_ke^{t_k}\right)^{n-2} \times p_ie^{t_i}\right.$$

$$\left. + \left(p_1e^{t_1} + p_2e^{t_2} + ... + p_ke^{t_k}\right)^{n-1} \times e^{t_i}\right]$$

$$\frac{d^2}{dt_i^{\,2}}M_x(t)\Big|_{t_i=0} = np_i\left[(n-1).\left(p_1 + p_2 + ... + p_k\right)^{n-2} \times p_i + (p_1 + p_2 + ... + p_k)^{n-1}\right.$$

$$= np_i\left[(n-1)p_i + 1\right]$$

$$= n(n-1)p_i^{\,2} + np_i$$

$$\mu_2' = n(n-1)p_i^{\,2} + np_i$$

$$\text{var}(x) = \mu_2' - \left(\mu_1'\right)^2$$

$$= n(n-1)p_i^{\,2} + np_i + \left(np_i\right)^2$$

$$= np_i[(n-1)p_i + 1 - np_i]$$

$$= np_i(1 - P_i)$$

$$= np_i q_i$$

$$V(x) = np_i q_i \, .$$

2.10 Discrete Uniform Distribution

Definition

A r.v X is said to have a discrete uniform distribution over the range [1, n] if its pmf is expressed as follows:

$$p(X = x) = \begin{cases} \frac{1}{n} & \text{for } x = 1, 2, ..., n \\ 0, & \text{otherwise} \end{cases},$$

here n is known as the parameter of the distribution and lies in the set of all positive integers. The above equation is also called a discrete rectangular distribution.

Such distribution can be conceived in practical if under the given experimental conditions, the different values of the random variable become equally likely. Thus for a die experiment and for an experiment with a deck of cards such distribution is appropriate.

To find Mean

$$E(X = i) = \Sigma x p(x)$$

$$= \Sigma i \frac{1}{n}$$

$$= \frac{1}{n} \sum_{i=1}^{n} x$$

$$= \frac{1}{n} \frac{n(n+1)}{2}$$

$$\text{Mean } = \frac{n+1}{2}$$

To find variance:

$$v(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \Sigma x^2 p(x)$$

$$= \Sigma i^2 \frac{1}{n}$$

$$= \frac{1}{n}\Sigma i^2 \qquad \because a_n = 1 + 2^2 + 3^2 + \ldots + n^2,\ S_n = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{n}\frac{n(n+1)(2n+1)}{6}$$

$$\therefore v(x) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n+1}{2}\left(\frac{(2n+1)}{3} - \frac{n+1}{2}\right)$$

$$= \frac{n+1}{2}\left(\frac{4n+2-3n-3}{6}\right)$$

$$= \frac{n+1}{2}\left(\frac{n-1}{6}\right)$$

$$v(x) = \frac{(n+1)(n-1)}{12}.$$

To find mgf

By the definition of mgf,

$$M_x(t) = E[e^{tx}]$$

$$= \Sigma e^{tx} p(x)$$

$$= \Sigma e^{tx}\frac{1}{n}$$

$$= \frac{1}{n}\sum_{x=1}^{n} e^{tx} \qquad \left(\because e^t + e^{2t} + e^{3t} + \ldots + e^{nt} = \frac{e^t(1-e^{nt})}{1-e^t}\right)$$

$$M_x(t) = \frac{1}{n}\frac{e^t(1-e^{nt})}{(1-e^t)}.$$

2.11 Fitting a Binomial and Poisson Distributions

Fitting a Binomial Distribution

When a binomial distribution is to be fitted to observed data, the following procedure is adopted:

1. Determine the values of p and q. If one of these values is known the other can be found out by the simple relationship p = (1-q) and q = (1-p). When p and q are equal, the distribution is symmetrical for p and q may be interchanged without alternating the value of any terms and consequently terms equidistant from the two ends of the series are equal. If p and q are unequal, the distribution is skew. If p is less than $\frac{1}{2}$, the distribution is positively skewed and when p is more than $\frac{1}{2}$ the distribution is negatively skewed.
2. Expand the binomial (p+q)n. The power n is equal to one less than the number of terms in the expanded binomial. Thus when two coins are tossed (n = 2), there will be three terms in the binomial. Similarly, when four coins are tossed (n=4) there will be five terms and so on.
3. Multiply each term of the expanded binomial by N (the total frequency) in order to obtain the expected frequency in each category.

The probability of 0, 1, 2, 3, … success would be obtained by the expansion of (p+q)n. Suppose this experiment is repeated for N times, then the frequency of r success is;

$$N \times P(r) = N \times \ {}^nC_r q^{n-r} p^r$$

Putting r = 0, 1, 2, …, n, we can get the expected of theoretical frequencies of the binomial distribution as follows:

Number of Success (r)	Expected or theoretical frequency (N P(r))
0	N qn
1	$N\ {}^{n}C_{1}q^{n-1}p$
2	$N\ {}^{n}C_{2}q^{n-2}p^{2}$
.	.
.	.
.	.
r	$N\ {}^{n}C_{r}q^{n-r}p^{r}$
n	N pn

Example:

8 coins are tossed at a time, 256 times. Find the expected frequencies of success (getting a head) and tabulate the result obtained

Solution:

$$p = \frac{1}{2}; q = \frac{1}{2;} n = 8; N = 256$$

The probability of success r times in n trials is given by $^{n}C_{r}q^{n-r}p^{r}$.

$$\therefore P(r) = ^{n} C_{r} q^{n-r} p^{r}$$

$$= ^{8} C_{r}\left(\frac{1}{2}\right)^{8-r}\left(\frac{1}{2}\right)^{r}$$

$$= ^{8} C_{r}\left(\frac{1}{2}\right)^{8}$$

Frequencies of 0, 1, 2, 3,…, 8 successes are:

Success	N P(r)	Expected frequency
0	$256\left(\frac{1}{256}\times\ {}^{8}C_{0}\right)$	1
1	$256\left(\frac{1}{256}\times\ {}^{8}C_{1}\right)$	8
2	$256\left(\frac{1}{256}\times\ {}^{8}C_{2}\right)$	28
3	$256\left(\frac{1}{256}\times\ {}^{8}C_{3}\right)$	56
4	$256\left(\frac{1}{256}\times\ {}^{8}C_{4}\right)$	70
5	$256\left(\frac{1}{256}\times\ {}^{8}C_{5}\right)$	56
6	$256\left(\frac{1}{256}\times\ {}^{8}C_{6}\right)$	28
7	$256\left(\frac{1}{256}\times\ {}^{8}C_{7}\right)$	8
8	$256\left(\frac{1}{256}\times\ {}^{8}C_{8}\right)$	1

Fitting a Poisson Distribution

When we want to fit a Poisson Distribution to a given frequency distribution, first we have to find out the arithmetic mean of the given data i.e., $\overline{X} = m$ when m is known the other values can be found out easily.

$$N\,P(X = x) = N \times \frac{e^{-\lambda}\lambda^{x}}{x!}; \qquad x = 0,1,2,\ldots,\infty.$$

$$NP(X = 0) = Ne^{-\lambda}$$

$$NP(X = 1) = NP(X = 0) \times \frac{m}{1}$$

$$NP(X = 2) = NP(X = 1) \times \frac{m}{2}$$

$$NP(X = 3) = NP(X = 2) \times \frac{m}{3}$$

$$NP(X = 4)) = NP(X = 3) \times \frac{m}{4} \text{ and so on.}$$

Example 1

100 Car Radios are inspected as they come of the production line and number of defects per set is recorded below:

No. of Defects	0	1	2	3	4
No. of sets	79	18	2	1	0

Fit a Poisson distribution to the above data and calculate the frequency of 0, 1, 2, 3 and 4 defects.

$\left(e^{-0.25} = 0.779\right)$

Solution

Fitting Poisson distribution

No. of Defectives (x)	No. of Sets (f)	(fx)
0	79	0
1	18	18
2	2	4
3	1	3
4	0	0
	N = 100	**$\sum$fx = 25**

$$\overline{X} = \frac{25}{100} = 0.25 = \lambda$$

$$e^{-25} = 0.779$$

$$N\,P(0) = N\,e^{-m} = 100 \times 0.779 = 77.90$$

$$N\,P(1) = N\,P(0) \times \frac{m}{1} = 77.90 \times 0.25 = 19.48$$

$$N\,P(2) = N\,P(1) \times \frac{m}{2} = 19.48 \times \frac{0.25}{2} = 2.44$$

$$N\,P(3) = N\,P(2) \times \frac{m}{3} = 2.44 \times \frac{0.25}{3} = 0.20$$

$$N\,P(4) = N\,P(3) \times \frac{m}{4} = 0.20 \times \frac{0.25}{4} = 0.10$$

Example 2

Fit a Poisson distribution to the following data and calculate the theoretical frequencies:

x:	0	1	2	3	4
f:	123	59	14	3	1

Solution

x	0	1	2	3	4	
f	123	59	14	3	1	$\sum f=200$
fx	0	59	28	9	4	$\sum fx = 100$

$\text{Mean} = \frac{100}{200} 0.25$

$NP_{(0)} = Ne^{-m}$

$= 200 \times e^{-0.5}$

$= 200 \times 6065 = 121.3$

Conclusion of expected frequencies:

x	Frequency N P(X=x)	
0	NP(0) =121.3	121
1	$NP(0) \times \frac{m}{1} = 121.3 \times 5 = 60.65$	61
2	$NP(1) \times \frac{m}{2} = \frac{60.65 \times 5}{2} = 15.16$	15
3	$NP(2) \times \frac{m}{3} = \frac{15.16 \times 5}{3} = 2.53$	3
4	$NP(3) \times \frac{m}{4} = \frac{2.53 \times 5}{4} = 0.29$	0
	Total	200

Exercise

1. Define one-point distribution.
2. Define Bernoulli distribution and derive its mean and variance.
3. For a binomial distribution mean is 6 and standard deviation is $\sqrt{2}$. Find the first two terms of the distribution.
4. Derive mgf of Binomial distribution and hence find its mean and variance.
5. Derive mgf of Poisson and hence derive its constants.
6. State and prove memory less property of geometric distribution.
7. Define multinomial distribution.

8. Derive the moment generating function and hence find its mean and variance of geometric distribution.
9. Define negative binomial distribution and derive its moment generating function and constants.
10. Bring out the relationship between binomial and Poisson distributions.
11. Define characteristic function. Also, state its properties.
12. Define moment generating function. Also, state and prove its any two properties.
13. Define discrete uniform distribution and derive its constants.

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Unit – III

Continuous Distributions

3.1 Introduction

In this unit, several parametric families of univariate probability density functions are presented. Also, mean, variance, moments, moments generating functions and characteristic functions of some continuous distributions are discussed elaborately.

3.2 Uniform Distribution (or Rectangular distribution)

Definition

A random variable X is said to have a continuous rectangular (uniform) distribution observes an interval (a, b)

i.e., $(-\infty < \mathrm{a} < \mathrm{b} < \infty)$, if its pdf is given by,

$$f(x;a,b)=\begin{cases}\dfrac{1}{b-a}, & if\ \ a<x<b\\ 0, & otherwise\end{cases}$$

Remarks:

1. a and b (a<b) are the two parameters of the distribution. The distribution is called uniform distribution over an interval (a, b) since it assumes a constant (uniform) value for all x in (a, b).
2. The distribution is also known as rectangular distribution, since the curve y = f(x) describes a rectangle over the x-axis and between the ordinates at x = a and x = b.
3. A uniform or rectangular variate X on the interval (a,b) is written as : X ~ U[a , b] or X ~ R[a, b]
4. The cumulative distribution function F(x) is given by:

$$F(x) = \begin{cases} 0 & , x \le a \\ \dfrac{x-a}{b-a} & , a < x < b \\ 1 & , x \ge b \end{cases}$$

Moments of Uniform Distribution

Let X ~ U[a,b]

$$\mu_r' = \int_a^b x^r f(x)dx$$

$$= \frac{1}{b-a}\int_a^b x^r dx$$

$$= \frac{1}{b-a}\left(\frac{b^{r+1} - a^{r+1}}{r+1}\right)$$

In particular,

$$\text{Mean} = E(x) = \mu_1' = \frac{1}{b-a}\left(\frac{b^2 - a^2}{2}\right) = \frac{b+a}{2}$$

$$\text{and } \mu_2' = \frac{1}{b-a}\left(\frac{b^3 - a^3}{3}\right) = \frac{1}{3}\left(b^2 + ab + a^2\right)$$

$$\therefore \text{ Variance= Var (x) } = \mu_2' - (\mu_1')^2$$

$$\therefore \operatorname{var}(x) = \frac{1}{3}\left(b^2 + ab + a^2\right) - \left\{\frac{1}{2}(b+a)\right\}^2$$

$$\text{var}(x) = \frac{1}{12}(b-a)^2.$$

MGF of Uniform distribution

$$M_x(t) = \int_a^b e^{tx} f(x)dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \int_a^b \frac{e^{tx}}{b-a} dx$$

$$= \frac{e^{bt} - e^{at}}{t(b-a)}, t \neq 0.$$

Characteristic function

$$\phi_x(t) = \int_a^b e^{itx} dx$$

$$= \frac{e^{ibt} - e^{iat}}{it(b-a)}, t \neq 0.$$

Mean Deviation about Mean (η)

$$\eta = E|X - Mean| = \int_a^b |X - Mean| f(x)dx$$

$$= \int_a^b \left| x - \frac{a+b}{2} \right| dx$$

$$= \frac{1}{b-a} \int_{-((b-a)/2}^{(b-a)/2} |t| dt \text{ where } t = x - \frac{a+b}{2}$$

$$= \frac{1}{b-a} 2 \int_0^{(b-a)/2} t\, dt = \frac{b-a}{4}$$

Example 1:

If X is uniformly distributed with mean 1 and variance 4/3, fine p(X<0).

Solution:

Let X ~ U[a,b],

So that $p(x) = \frac{1}{b-a}, a < x < b$

Given mean = 1 and variance = $\frac{4}{3}$.

$\Rightarrow$ Mean $= \frac{1}{2}(b+a) = 1 \Rightarrow b + a = 2$ and $Var(x) = \frac{1}{12}(b-a)^2 = \frac{4}{3}$

$$\Rightarrow (b-a)^2 = \frac{48}{3}$$

$$\Rightarrow (b-a)^2 = 16$$

$$\Rightarrow b - a = \pm 4$$

Solving, we get a = -1 and b = 3; (a<b),

$$\therefore p(x) = \frac{1}{4}; -1 < x < 3$$

$$P(X < 0) = \int_{-1}^{0} p(x)dx$$

$$= \frac{1}{4}[x]_{-1}^{0} = \frac{1}{4}$$

Example 2:

If X has a uniform distribution in [0, 1], find the distribution of -2 log X. Identify the distribution also.

Solution:

Let Y = - 2 log X.

Then the distribution function F of Y is given by:

$$G_Y(y) = P(Y \le y) = (-2\log X \le Y)$$

$$= P\left(\log X \ge -\frac{y}{2}\right) = P\left(X \ge e^{-\frac{y}{2}}\right)$$

$$= 1 - P\left(X < e^{-\frac{y}{2}}\right)$$

$$= 1 - \int_0^{e^{-\frac{y}{2}}} f(x)dx$$

$$= 1 - \int_0^{e^{-\frac{y}{2}}} 1.dx = 1 - e^{-\frac{y}{2}}$$

$$G_Y(y) = \frac{d}{dy}G(y) = \frac{1}{2}e^{-\frac{y}{2}}, 0 < y < \infty .$$

3.3 Normal Distribution or Gaussian Distribution

A random variable X is said to follow a normal distribution if its pdf is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma > 0$$

Here, f(x) is a legitimate density function as the total area under the normal curve is unity.

To prove that total probability is one,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx$$

$$\text{put } t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$dt = \frac{1}{\sqrt{2}\sigma}\, dx$$

$$\Rightarrow dx = \sqrt{2}\sigma\, dt$$

$$= \int_{\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2} \sqrt{2}\,\sigma\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$\text{put } t^2 = y$$

$$\Rightarrow \quad t = \sqrt{y}$$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y} \frac{1}{2\sqrt{y}}\, dy$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} e^{-y} y^{-1/2}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} y^{\frac{1}{2}-1} e^{-y} dy$$

We know that $\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$

$$= \frac{1}{\sqrt{\pi}}\, \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}} \sqrt{\pi}$$

$$=1$$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = 1$$

f(x) is a legitimate density function.

Mean and Variance of N(μ,σ²)

If $X \sim N(\mu,\sigma^2)$, then $E(X) = \mu$ and $V(X) = \sigma^2$.

Proof

$$E(X) = \int_{\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put

$$t = \frac{x-\mu}{\sqrt{2}\sigma} \Rightarrow x = \mu + \sqrt{2}\,\sigma\, t$$

$$dt = \frac{1}{\sqrt{2}\sigma} dx$$

$$dx = \sqrt{2}\sigma\, dt$$

$$= \int_{-\infty}^{\infty} \left(\mu + \sqrt{2}\,\sigma\, t\right) \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2} \sqrt{2}\,\sigma\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\mu + \sqrt{2}\,\sigma\, t\right) e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mu\, e^{-t^2} dt + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sqrt{2}\sigma\, t\, e^{-t^2} dt$$

$$=\frac{\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}dt+\frac{\sqrt{2}\sigma}{\sqrt{\pi}}\int_{-\infty}^{\infty} t\,e^{-t^2}dt$$

$$=\frac{\mu}{\sqrt{\pi}}\times\sqrt{\pi}+\frac{\sqrt{2}\sigma}{\sqrt{\pi}}\times(0)=\mu=\mu_1'$$

$\therefore$ Mean = E(X) = μ

To find variance,

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)dx$$

$$=\int_{-\infty}^{\infty} x^2\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

Put

$$t=\frac{x-\mu}{\sqrt{2}\sigma}\Rightarrow x=\mu+\sqrt{2}\,\sigma\,t$$

$$dt=\frac{1}{\sqrt{2}\sigma}dx$$

$$\Rightarrow dx=\sqrt{2}\,\sigma\,dt$$

$$=\int_{-\infty}^{\infty}\left(\mu+\sqrt{2}\,\sigma\,t\right)^2\frac{1}{\sigma\sqrt{2\pi}}e^{-t^2}\sqrt{2}\,\sigma\,dt$$

$$=\int_{-\infty}^{\infty}\left(\mu^2+2\sigma^2t^2+2\mu\sqrt{2}\,\sigma\,t\right)\frac{1}{\sigma\sqrt{2\pi}}e^{-t^2}\sqrt{2}\,\sigma\,dt$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mu^2e^{-t^2}dt+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}2\sigma^2t^2e^{-t^2}dt+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}2\mu\sqrt{2}\,\sigma\,t\,e^{-t^2}dt$$

$$=\frac{\mu^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}dt+\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}t^2e^{-t^2}dt+\frac{1}{\sqrt{\pi}}2\sqrt{2}\mu\sigma\int_{-\infty}^{\infty}t\,e^{-t^2}dt$$

$$=\frac{\mu^2}{\sqrt{\pi}}\times\sqrt{\pi}+\frac{2\sigma^2}{\sqrt{\pi}}\times2\int_{0}^{\infty}t^2e^{-t^2}dt+0$$

Put $t^2 = y$

$2t\,dt = dy$

$$\Rightarrow dt = \frac{dy}{2\sqrt{y}}$$

$$\therefore E\left(X^2\right) = \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \times 2\int_0^\infty e^{-y}\, y\, \frac{dy}{2\sqrt{y}}$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-y} \sqrt{y}\, dy$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty y^{1/2} e^{-y} dy$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty y^{3/2-1} e^{-y} dy$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(3/2\right)$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2}$$

$$\therefore \mu_2' = \mu^2 + \sigma^2.$$

$$\therefore V(X) = \mu_2' - \left(\mu_1'\right)^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$\therefore Var(X) = \sigma^2.$$

Standard Normal Variate or Standard Norman Distribution

If X follows normal distribution $N(\mu, \sigma^2)$,then $z = \frac{x-\mu}{\sigma}$ is a standard normal variate with mean zero and variance one and is denoted by N(0,1).

The pdf of standard normal variate is given by,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x}{2}}; \quad -\infty < x < \infty$$

MGF and Mean and Variance

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put

$$z = \frac{x-\mu}{\sigma} \Rightarrow x = z\sigma + \mu$$

$$dz = \frac{1}{\sigma} dx$$

$$\Rightarrow dx = \sigma\,dz$$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+z\sigma)} e^{-\frac{z^2}{2}} \sigma\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+z\sigma)} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} e^{t\sigma z - \frac{z^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2 - 2t\sigma z\right)} dz$$

Add and subtract by $\sigma^2 t^2$

$$=\frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2-2t\sigma z+\sigma^2 t^2-\sigma^2 t^2\right)}dz$$

$$=\frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(z-\sigma t)^2-\sigma^2 t^2\right]}dz$$

$$=\frac{e^{\mu t}}{\sqrt{2\pi}}\frac{\sigma^2 t^2}{e^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2}dz$$

Put

U= z- σt

du = dz

$$=\frac{e^{\mu t}e^{\sigma^2\frac{t^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}du$$

$$=e^{\mu t}\,e^{\sigma^2\frac{t^2}{2}}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}du$$

($\because$ the total probability of Standard Normal is one)

$$M_X(t)=e^{\mu t+\sigma^2\frac{t^2}{2}}$$

To find Mean and Variance

$$M_X(t)=e^{\mu t+\sigma^2\frac{t^2}{2}}$$

$$= e^{t\left(\mu+\frac{\sigma^2 t}{2}\right)}$$

$$=1+t\left(\mu+\frac{\sigma^2 t}{2}\right)+\frac{t^2}{2!}\left(\mu+\frac{\sigma^2 t}{2}\right)^2+\frac{t^3}{3!}\left(\mu+\frac{\sigma^2 t}{2}\right)^3+\ldots$$

$$=1+\frac{t}{1!}\mu+\frac{t^2}{2!}\sigma^2+\frac{t^2}{2!}\mu^2+\ldots.$$

The coefficient of $\frac{t}{1!} = \mu = \mu_1'$

$\therefore$ Mean = μ.

The coefficient of $\frac{t^2}{2!}$ *is* $\sigma^2 + \mu^2$.

$$\therefore \mu_2' = \sigma^2 + \mu^2$$

$$\therefore Var(X) = \mu_2' - \left(\mu_1'\right)^2$$

$$= \sigma^2 + \mu^2 - \mu^2$$

$$\therefore Var(X) = \sigma^2$$

The first four Moments about Origin

$$M_X(t) = e^{\mu t + \sigma^2 \frac{t^2}{2}}$$

$$= e^{t\left(\mu + \sigma^2 \frac{t}{2}\right)}$$

$$= 1 + t\left(\mu + \frac{\sigma^2 t}{2}\right) + \frac{t^2}{2!}\left(\mu + \frac{\sigma^2 t}{2}\right)^2 + \frac{t^3}{3!}\left(\mu + \frac{\sigma^2 t}{2}\right)^3 + \frac{t^4}{4!}\left(\mu + \frac{\sigma^2 t}{2}\right)^4 + \ldots$$

$$= 1 + \frac{t}{1!}\mu + \frac{t^2}{2!}\sigma^2 + \frac{t^2}{2!}\left(\mu^2 + \mu\sigma^2 t + \sigma^4 \frac{t^2}{4}\right)$$

$$+ \frac{t^3}{3!}\left(\mu^3 + 3\mu^2\sigma^2 \frac{t}{2} + 3\mu \frac{\sigma^4 t^2}{4} + \frac{\sigma^6 t^3}{8}\right)$$

$$+ \frac{t^4}{4!}\left(\mu^4 + 4\mu^3 \frac{\sigma^2 t}{2} + 6\mu^2\left(\frac{\sigma^2 t}{2}\right)^2 + 4\mu\left(\frac{\sigma^2 t}{2}\right)^3 + \left(\frac{\sigma^2 t}{2}\right)^4\right) + \ldots$$

$$\therefore \mu_1' = \text{The Coefficient of } \frac{t}{1!} = \mu$$

$$\mu_2' = The\,Coefficient\,of\ \frac{t^2}{2!} = \sigma^2 + \mu^2$$

$$\mu_3' = The\,Coefficient\,of\ \frac{t^3}{3!} = 3\mu\sigma^2 + \mu^3$$

$$\mu_4' = The\,Coefficient\,of\ \frac{t^4}{4!} = 3\sigma^2 + 6\mu^2\sigma^2 + \mu^4$$

The First Four Central Moments

We know that, $\mu_0 = 1, \mu_1 = 0$

By the definition of central moments,

$$\mu_r = E(X-\mu)^r$$

$$\therefore \mu_2 = E(X-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put

$$t = \frac{x-\mu}{\sqrt{2}\,\sigma}$$

$$\Rightarrow x-\mu = \sqrt{2}\,\sigma t \quad , x = \sqrt{2}\,\sigma t + \mu$$

$$\Rightarrow dt = \frac{1}{\sqrt{2}\,\sigma} dx$$

$$= \int_{-\infty}^{\infty} \left(\sqrt{2}\,\sigma t\right)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2} \sqrt{2}\,\sigma\, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} t^2\, dt$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\,2\int_0^\infty t^2 e^{-t^2}\,dt$$

Put $t^2 = y \quad \Rightarrow 2t\,dt = dy$

$$\Rightarrow dt = \frac{1}{2\sqrt{y}}\,dy$$

$$=\frac{4\sigma^2}{\sqrt{\pi}}\int_0^\infty y\,e^{-y}\,\frac{1}{2\sqrt{y}}\,dy$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\int_0^\infty y^{1-\frac{1}{2}}e^{-y}\,dy$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\int_0^\infty y^{\frac{1}{2}}e^{-y}\,dy$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\int_0^\infty y^{3-\frac{1}{2}}e^{-y}\,dy$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}$$

$$\mu_2 = \sigma^2$$

(or)

$$\mu_2 = \mu_2{}' - (\mu_1{}')^2$$

$$= \sigma^2 + \mu^2 - \mu^2$$

$$\mu_2 = \sigma^2$$

$$\mu_3 = \mu_3{}' - 3\mu_2{}'\mu_1{}' + 2(\mu_1{}')^3$$

$$= 3\mu\sigma^2 + \mu^3 - 3(\sigma^2 + \mu^2)\mu + 2\mu^3 = 0$$

$\therefore \mu_3 = 0$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4 - 4(3\mu\sigma^2 + \mu^3)\mu + 6(\sigma^2 + \mu^2)\mu^2 - 3\mu^4$$

$$= 3\sigma^4 + 12\mu^2\sigma^2 - 12\mu^2\sigma^2 + 7\mu^4 - 7\mu^4$$

$\therefore \mu_4 = 3\sigma^4$

The r

If X is a normal variate then the all odd order central moments does not exists, but all even order central moments exists.

Proof

By the definition of r^{th} order central moment

$$\mu_r = E(X-\mu)^r$$

$$= \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^r \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\text{Put } t = \frac{x-\mu}{\sqrt{2}\,\sigma}$$

$$\Rightarrow x-\mu = \sqrt{2}\,\sigma t \quad , \; x = \sqrt{2}\,\sigma t + \mu$$

$$\Rightarrow dt = \frac{dx}{\sqrt{2}\,\sigma}$$

$$\Rightarrow dx = dt\sqrt{2}\,\sigma$$

$$\mu_r = \int_{-\infty}^{\infty} (\sqrt{2}\,\sigma t)^r \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2} \sqrt{2}\,\sigma\, dt$$

$$\mu_r = \frac{(2)^{\frac{r}{2}} \sigma^r}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^r e^{-t^2} dt \qquad (1)$$

Case (i)

If r is an odd integer, r = 2n+1.

From the equation (1),

$$\mu_{2n+1} = \frac{2^{\frac{2n+1}{2}} \sigma^{2n+1}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^{2n+1} e^{-t^2} dt$$

$$\mu_{2n+1} = 0, \quad n = 0,1,2,....\left(\because t^{2n+1} e^{-t^2} \text{ is an odd function}\right)$$

$$\mu_1 = \mu_3 = \mu_5 = ... = 0$$

Case (ii)

If r is an even integer, r = 2n.

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^{2n} e^{-t^2} dt$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} 2\int_{0}^{\infty} t^{2n} e^{-t^2} dt$$

$$\text{Put } y = t^2 \quad \Rightarrow t = \sqrt{y} = y^{\frac{1}{2}}$$

$$dy = 2t\, dt$$

$$\Rightarrow dt = \frac{1}{2\sqrt{y}} dy$$

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} 2\int_{0}^{\infty} y^{\frac{2n}{2}} e^{-y} \frac{1}{2\sqrt{y}} dy$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_{0}^{\infty} y^{n-\frac{1}{2}} e^{-y} dy$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^\infty y^{\left(n+\frac{1}{2}\right)-1} e^{-y}\, dy$$

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) \qquad (2)$$

After simplification, we get,

$$\mu_{2n} = 1.3.5.7....(2n-1).\sigma^2 n \qquad (3)$$

when $n = 1, \mu_2 = 1.\sigma^{2(1)} = \sigma^2$

when $n = 2, \mu_4 = 3.\sigma^{2(2)} = 3\sigma^4$

and so on.

The Recurrence relations of Central Moments

We consider the equation (2),

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)$$

Put n = n-1, 2n = 2(n-1) = 2n-2

Also,

$$\mu_{2n-2} = \frac{2^{n-1} \sigma^{2(n-1)}}{\sqrt{\pi}} \Gamma\left(n - 1 + \frac{1}{2}\right)$$

$$\mu_{2n-2} = \frac{2^{n-1} \sigma^{2n-2}}{\sqrt{\pi}} \Gamma\left(n - \frac{1}{2}\right) \qquad (4)$$

From the equations (2) and (4), we get,

$$\frac{\mu_{2n}}{\mu_{2n-2}} = \frac{\frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)}{\frac{2^{n-1} \sigma^{2(n-1)}}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)}$$

$$= \frac{2\sigma^2\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)}{\Gamma\left(n-\frac{1}{2}\right)}$$

$$\frac{\mu_{2n}}{\mu_{2n-2}} = 2\sigma^2\,\frac{2n-1}{2}$$

$$\frac{\mu_{2n}}{\mu_{2n-2}} = (2n-1)\,\sigma^2$$

$$\Rightarrow \mu_{2n} = (2n-1)\sigma^2\,\mu_{2n-2}$$

which is the recurrence relation of the even order central moment of normal distribution.

Additive Property (or) Reproductive Property:

If $X_1, X_2, \ldots X_n$ are n independent normal variates with mean $\mu_1, \mu_2, \ldots \mu_n$ and variance $\sigma_1^{\,2}, \sigma_2^{\,2}, \ldots, \sigma_n^{\,2}$ respectively,then $\sum_{i=1}^{n} a_i x_i$ is also a normal variate with mean $\sum_{i=1}^{n} a_i \mu_i$ and variance $\sum_{i=1}^{n} a_i \sigma_i^{\,2}$.

Proof

The mgf of normal distribution is,

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$\Rightarrow M_{\sum_{i=1}^{n} a_i x_i}(t) = M_{a_1 x_1}(t) . M_{a_2 x_2}(t) \ldots\ldots M_{a_n x_n}(t)$$

$$= e^{a_1 \mu_1 t + \frac{a_1^2 \sigma_1^2 t^2}{2}} . e^{a_2 \mu_2 t + \frac{a_2^2 \sigma_2^2 t^2}{2}} \ldots\ldots$$

$$M_{\sum_{i=1}^{n} a_i x_i}(t) = e^{\sum_{i=1}^{n} a_i \mu_i t + \sum_{i=1}^{n} \frac{a_i^2 \sigma_i^2 t^2}{2}}$$

Which is the mgf of normal distribution with mean $\sum a_i x_i$ and variance $\sum a_i \sigma_i^{\,2}$.

3.4 Cauchy Distribution

Let us consider a roulette wheel in which the probability of the pointer stopping at any part of the circumference is constant. In other words, the probability that any value of θ lies in the interval [-π/2, π/2] is constant and consequently θ is a rectangular variate in the range [-π/2, π/2] with probability differential given by:

$$dP(\theta) = \begin{cases} (1/\pi)\ d\theta, & -\pi/2 \le \theta \le \pi/2 \\ 0, & otherwise \end{cases}$$

Let us now transform to variable X by the substitution : x = r tanθ $\Rightarrow dx = r\sec^2\theta d\theta$.

Since $-\pi/2 \le \theta \le \pi/2$, the range for X is from $-\infty$ to ∞. Thus the probability differential of X becomes:

$$dF(x) = \frac{1}{\pi}.\frac{dx}{r\sec^2\theta}$$

$$= \frac{1}{\pi}.\frac{dx}{\left[r\left\{1+\left(x^2/r^2\right)\right\}\right]}$$

$$= \frac{r}{\pi}.\frac{dx}{x^2+r^2}; \qquad -\infty < x < \infty$$

In particular if we take r = 1, we get, $f(x) = \frac{1}{\pi}.\frac{1}{1+x^2}; \qquad -\infty < x < \infty$

This is the pdf of a standard Cauchy variate and we write $X \sim C(1,0)$.

Definition:

A random variable X is said to have a standard Cauchy distribution if its pdf is given by:

$$f_x(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty \qquad \text{.....................(1)}$$

and X is termed as standard Cauchy variate.

More generally, Cauchy distribution with parameter λ and μ has the pdf.,

$$g_y(y) = \frac{\lambda}{\pi\left[\lambda^2+(y-\mu)^2\right]}, -\infty < y < \infty; \lambda > 0 \qquad \text{.....................(2)}$$

and write X ~ C(λ, μ). But putting X = (Y - μ)/λ in (2), we get (1),hence if Y ~ C(λ, μ) then X = (Y - μ)/λ ~ C(1,0).

Characteristic function of Cauchy distribution

If X is a standard Cauchy variate then

$$\phi_X(t) = E\left[e^{itx}\right]$$

$$= \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

$$\phi_x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

To evaluate (3), consider standard Laplace distribution

$$f_1(z) = \frac{1}{2} e^{-|z|}, -\infty < z < \infty$$

Then $\phi_x(t) = \phi_z(t) = E(e^{itz}) = \dfrac{1}{1+t^2}$

Since $\phi_1(t)$ is absolutely integrable in $(-\infty, \infty)$, we have by Inversion theorem,

$$\frac{1}{2} e^{-|z|} = f_1(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itz} \phi_1(t) dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itz}}{1+t^2} dt$$

$$\Rightarrow \quad e^{-|z|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{itz}}{1+t^2} dt \qquad \text{[changing t to –t]}$$

On interchanging t and z, we have $\Rightarrow \quad e^{-|t|} = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{e^{itz}}{1+t^2} dz$ ……….(4)

From (3) and (4) we get, $\phi_x(t) = e^{-|t|}$.

Additive property of Cauchy distribution

If X_1 and X_2 are independent Cauchy variateswith parameters (λ_1, μ_1) and (λ_2, μ_2) respectively, then $X_1 + X_2$ is Cauchy variate with parameters $((\lambda_1+\lambda_2, \mu_1+ \mu_2)$.

Proof:

$$\phi_x(t) = \exp\{i\mu_j t - \lambda_j |t|\}, (j = 1,2)$$

$$\phi_{x_1+x_2}(t) = \phi_{x_1}(t)\phi_{x_2}(t)$$

$$= \exp[it(\mu_1 + \mu_2) - (\lambda_1 + \lambda_2)|t|]$$

and the result follows by uniqueness theorem of characteristic functions.

Moments of Cauchy distribution

$$E(y) = \int_{-\infty}^{\infty} y(f(y)dy$$

$$= \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y}{\lambda^2 + (y-\mu)^2} dy$$

$$= \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{(y-\mu)+\mu}{\lambda^2 + (y-\mu)^2} dy$$

$$= \mu\frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{dy}{\lambda^2 + (y-\mu)^2} + \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{(y-\mu)}{\lambda^2 + (y-\mu)^2} dy$$

$$= \mu.1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{z}{\lambda^2 + z^2} dz$$

Although the integral $= \int_{-\infty}^{\infty} \frac{z}{\lambda^2 + z^2} dz$, is not completely convergent, that is

$\lim_{\substack{n\to\infty \\ n'\to\infty}} \int_{-n}^{n'} \frac{z}{\lambda^2 + z^2} dz$, does not exist, its principal value, viz, $\lim_{n\to\infty} \int_{-n}^{n} \frac{z}{\lambda^2 + z^2} dz$, exists and is equal to zero. Thus, in the general, the mean of Cauchy distribution does not exist. But, if we conventionally agree to assume that the mean of Cauchy distribution exists (by taking the principal value), then it is located at $x = \mu$. Also, obviously, the probability curve is symmetrical about the point x = μ. Hence for this distribution, the mean, median, mode coincide at the point x = μ.

$$\mu_2 = E(Y-\mu)^2 = \int_{-\infty}^{\infty} (y-\mu)^2 f(y)dy = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{(y-\mu)^2}{\lambda+(y-\mu)^2}dy,$$

which does not exist since the integral is not convergent. Thus, in general, for the Cauchy's distribution the moment μ, $(r \geq 2)$ do not exist.

Remark:

The role of Cauchy distribution in statistical theory often lies in providing counter examples, e.g., it is often quoted as a distribution for which moments do not exist. It also provides an example to show that $\phi_{x_1+x_2}(t) = \phi_{x_1}(t)\phi_{x_2}(t)$ does not imply that X and Y are independent.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a standard Cauchy distribution. Let $\overline{X} = \sum_{i=1}^{n} X_x / n.$ Since $E(X_i)$ does not exist. $E(\overline{X})$ does not exist and the definition of an unbiased estimate does not apply to $\overline{X}$, Cauchy distribution also contradicts the weak law of large numbers.

Example 1:

Let X have a standard Cauchy distribution. Find a pdf for X^2 and identify its distribution.

Solution:

Let X has a standard Cauchy distribution, its pdf is:

$$f_x(x) = \frac{1}{\pi(1+x^2)}, \ -\infty < x < \infty$$

The distribution function F(.) of $Y = X^2$ is:

$$G_y(y) = P(Y \leq y) = P(X^2 \leq y) = p\left(-\sqrt{y} \leq X \leq \sqrt{y}\right)$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx = 2\frac{1}{\pi}\int_{0}^{\sqrt{y}} \frac{dx}{1+x^2} = \frac{2}{\pi}\tan^{-1}\left(\sqrt{y}\right), 0 < y < \infty$$

The pdf $g_x(y)$ of Y is given by:

$$g_y(y) = \frac{d}{dy}\left[G_y(y)\right]$$

$$= \frac{2}{\pi} \cdot \frac{1}{(1+y)} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\pi} \cdot \frac{y^{-1/2}}{1+y}$$

$$= \frac{1}{B\left(\frac{1}{2}, \frac{1}{2}\right)} \cdot \frac{y^{\frac{1}{2}-1}}{(1+y)^{\frac{1}{2}+\frac{1}{2}}} \text{ , } y > 0$$

This is the pdf of Beta distribution of second king with parameter $\left(\frac{1}{2}, \frac{1}{2}\right)$

i.e., $X^2 \sim \beta\left(\frac{1}{2}, \frac{1}{2}\right)$.

3.5 Log Normal Distribution

Definition

The positive random variable X is said to have log normal distribution if $\log_e X$ is normally distributed. Let $Y = \log_e X$ is normally distributed.

Let X be a positive random variable and let a new random variable $Y = \log_e X$. If Y has a normal distribution, then X is said to have a log normal distribution.

For x > 0:

The cdf is

$$F(x) = \Pr(X \leq x)$$

$$= \Pr(\log X \leq \log x)$$

$$= \Pr(Y \leq \log x)$$

$$= \int_{-\infty}^{\log x} f(y)dy$$

$$= \int_{-\infty}^{\log x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

Let $y = \log_e^u$

$dy = \frac{1}{u} du$

$y = -\infty, u = 0$

$y = \log x, u = x$

$$\therefore F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}\left(\frac{\log u-\mu}{\sigma}\right)^2} .\frac{1}{u} du$$

$$F(x) = \int_0^x \frac{1}{u\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log u-\mu}{\sigma}\right)^2} .\frac{1}{u} du$$

For $u \le 0, f(x) = 0$

$$\therefore f(x) = \begin{cases} \frac{1}{u\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log u-\mu}{\sigma}\right)^2} & ; u > 0 \\ 0 & ; u \le 0 \end{cases}$$

Which is the pdf of log normal distribution.

Note:

If $x \sim N(\mu, \sigma^2)$,then $y = e^x$ is called lognormal random variable. Since its logarithm $\log y = \log e^x = X$ is a normal random variable.

Moments and hence mean & Variance

By the definition of rth moments about origin,

$$\mu^r = E(X^r)$$

$$= E\left[\left(e^y\right)^r\right] \qquad \text{y= log x}$$

$$= \text{E}[\text{e}^{\text{yr}}]$$

$$= M_Y(r)$$

$$\therefore \mu_r' = e^{\mu_r + \frac{\sigma^2 r^2}{2}} \qquad (1)$$

To find mean and variance:

Put r = 1 in (1), we get

$$\mu_1' = e^{\mu + \frac{\sigma^2}{2}} = mean$$

Put r = 2,

$$\mu_2' = e^{2\mu + \frac{4\sigma^2}{2}}$$

$$\mu_2' = e^{2(\mu + \sigma^2)}$$

$$\therefore \text{var}(X) = \mu_2' - (\mu_1')^2$$

$$= e^{2(\mu + \sigma^2)} + e^{2\left(\mu + \frac{\sigma^2}{2}\right)}$$

$$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$\text{Var(X)} = e^{2\mu + \sigma^2}\left(e^{\sigma^2 - 1}\right).$$

Exercise

1. Define uniform distribution and derive its mean and variance.
2. Draw the curve for the distribution function of the normal distribution.
3. Define Normal distribution. Derive its MGF, Mean and Variance. Also, Derive its first four moments about origin.
4. Define Cauchy distribution. Give its expectation value.
5. Define lognormal distribution and derive its constants.
6. Derive mgf of lognormal distribution and hence find its mean and variance.

Unit – IV

Continuous Distributions (Continuation)

4.1 Introduction

In this unit, several parametric families of univariate probability distributions are presented. Also, mean, variance, moments, moments generating functions and characteristic functions of some continuous distributions are elaborately explained.

4.2 Exponential Distribution

The exponential distribution has been used as a model for lifetimes of various things. The length of the time interval between successive happenings can be shown to have an exponential distribution, provided that the number of happening in a fixed time interval has a Poisson distribution.

(i) Exponential is a special case of the Gamma distribution.

(ii) Also, sum of independently identically distributed exponential random variables is gamma distribution.

Definition:

A continuous r.v X is said to follow a exponential distribution with parameter λ>0 if its pdf is given by,

$$f(x) = \lambda e^{-\lambda x}, \ \ where \ 0 \le x < \infty$$

Moments, Mean and Variance

By the definition of r^th^ moments about origin,

$$\mu_r' = E(X^r)$$

$$= \int_0^\infty x^r f(x)\, dx$$

$$= \int_0^\infty x^r \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x^r e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x^{(r+1)-1} e^{-\lambda x}\, dx$$

$$= \lambda \ \frac{\Gamma(r+1)}{\lambda^{r+1}} \ \left(\because \int_0^\infty x^{n-1} e^{-ax} dx = \frac{\Gamma n}{a^n} \right)$$

$$= \lambda \frac{(r+1-1)!}{\lambda^{r+1}}$$

$$= \lambda \frac{r!}{\lambda^{r+1}}$$

$\mu_r' = \frac{r!}{\lambda^r}$ which is the r^{th} raw moments of Exponential distribution.

Put r=1; $\mu_1' = \frac{1}{\lambda}$, Mean= $\frac{1}{\lambda}$.

Put r=2; $\mu_2' = \frac{2!}{\lambda^2}$

$$\mathrm{Var}(x) = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{2!}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\mathrm{Var}(x) = \frac{1}{\lambda^2}.$$

MGF and hence Mean and Variance

By the definition of mgf,

$$M_X(t) = E\left[e^{tx}\right]$$

$$= \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{tx} e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)}\right]_0^\infty$$

$$= \lambda \left[0 + \frac{1}{\lambda - t}\right]$$

$M_X(t) = \dfrac{\lambda}{\lambda - t}$ which is the mgf of exponential distribution.

To find Mean and Variance:

$$\because M_X(t) = \frac{\lambda}{\lambda - t}$$

$$= \frac{1}{1 - \frac{t}{\lambda}}$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \ldots + \frac{t^r}{\lambda^r}$$

$$= 1 + \frac{1}{\lambda}\frac{t}{1!} + \frac{2!}{\lambda^2}\frac{t^2}{2!} + \frac{3!}{\lambda^3}\frac{t^3}{3!} + \ldots + \frac{r!}{\lambda^r}\frac{t^r}{r!}$$

The Coefficient of $\frac{t}{1!} is \frac{1}{\lambda} = \mu_1{}' = Mean$

The Coefficient of $\frac{t^2}{2!} is \frac{2}{\lambda^2} = \mu_2{}'$

Therefore, variance $\mu_2 = \mu_2{}' - \left(\mu_1{}'\right)^2$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\therefore Mean = \frac{1}{\lambda} \; and \; V(X) = \frac{1}{\lambda^2}$$

Memory less Property of Exponential Distribution

Statement

Let X be exponential distributed r.v with parameter λ. Then any two positive integer m and n, $P_r(X > m + n \mid X > m) = P(X > n)$.

That is, Let X be the life time of a given component, then the conditional probability that the corresponding until last m+n time units given that, it has lasted m time units, is same as initial probability of lasted n time units.

Another way, we can say that an "old" functioning component has the some life time distribution as a "new" functioning components or that the component is not subject to fatigue or to wear.

Proof

$$\text{LHS} \Rightarrow P_r(X > m+n \mid x > m) = \frac{P(X > m+n \cap X > m)}{P(X > m)}$$

$$= \frac{P(X > m+n)}{P(X > m)}$$

Given X is an exponentially distributed r.v

$$\therefore f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

For any k,

$$P_r(X > k) = \int_k^\infty f(x)\,dx$$

$$= \int_k^\infty \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_k^\infty e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^\infty$$

$$= \left[-e^{-\lambda x}\right]_k^\infty$$

$$P_r(X > k) = e^{-\lambda k}$$

$$P(X > m+n) = e^{-\lambda(m+n)}$$

$$P(X > m) = e^{-\lambda m}$$

$$P(X > n) = e^{-\lambda n}$$

$$P_r(X > m+n \mid x > m) = \frac{P(X > m+n)}{P(X > m)}$$

$$= \frac{e^{-\lambda m - \lambda n}}{e^{-\lambda m}}$$

$$= e^{-\lambda n}$$

$$P_r(X > m+n \mid x > m) = P_r(X > n).$$

Median of Exponential Distribution

The median is defined as the value of the variable which divides the total area into two equal parts. The median is defined is,

$$\int_0^{med} f(x)\, dx = \frac{1}{2} \ (or) \ \int_{med}^{\infty} f(x)\, dx = \frac{1}{2}$$

Consider, $\int_0^{md} f(x)\, dx = \frac{1}{2}$

$$\Rightarrow \int_0^{md} \lambda e^{-\lambda x}\, dx = \frac{1}{2}$$

$$\Rightarrow \lambda \int_0^{md} e^{-\lambda x}\, dx = \frac{1}{2}$$

$$\Rightarrow \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{md} = \frac{1}{2}$$

$$\Rightarrow \left[-e^{-md\lambda} + 1 \right] = \frac{1}{2}$$

$$\Rightarrow -e^{-md\,\lambda} = \frac{1}{2} - 1$$

$$\Rightarrow e^{-md\,\lambda} = \frac{1}{2}$$

$$\Rightarrow -md\,\lambda = \log\left(\frac{1}{2}\right)$$

$$\Rightarrow -\lambda\, md = \log 1 - \log 2$$

$$\Rightarrow -\lambda\, md = -\log 2$$

$$\Rightarrow md = \frac{1}{\lambda} \log 2 .$$

4.3 Gamma Distribution (Two Parameter form or Erlang Distribution)

Definition

A continuous random variable X is said to Erlang distribution or Gamma distribution $\lambda>0$ and $k>0$ if its pdf is given by,

$$f(x) = \begin{cases} \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{where } \Gamma(k) = \int_0^\infty x^{k-1} e^{-x}\, dx$$

Note:

Prove that the total probability of Gamma distribution is 1.

To prove $\int_0^\infty f(x)\, dx = 1$

$$\therefore LHS = \int_0^\infty f(x)\, dx = \int_0^\infty \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}\, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \frac{\Gamma(k)}{\lambda^k}$$

$$= 1 .$$

MGF and hence Mean and Variance of Gamma Distribution

By the definition of mgf,

$$M_X(t) = E\left[e^{tx}\right]$$

$$= \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$

$$= \frac{\lambda^k}{\Gamma(k)} \frac{\Gamma(k)}{(\lambda - t)^k} \quad \left(\because \int_0^\infty x^{n-1} e^{-ax} dx = \frac{\Gamma(n)}{a^n} \right)$$

$$= \left(\frac{\lambda}{\lambda - t} \right)^k \quad = \frac{1}{\left(1 - \frac{t}{\lambda}\right)^k}$$

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-k}.$$

To find Mean and Variance

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-k}$$

$$= 1 + \frac{k}{1!}\frac{t}{\lambda} + \frac{k(k+1)}{2!}\left(\frac{t}{\lambda}\right)^2 + \ldots$$

$$= 1 + \frac{k}{\lambda}\frac{t}{1!} + \frac{k(k+1)}{\lambda^2}\frac{t^2}{2!} + \ldots$$

The Coefficient of $\frac{t}{1!} = \frac{k}{\lambda} = \mu_1{}' = Mean$

The Coefficient of $\frac{t^2}{2!} = \frac{k(k+1)}{\lambda^2} = \mu_2{}'$

$$\therefore Var(X) = \mu_2 = \mu_2{}' - \left(\mu_1{}'\right)^2$$

$$= \frac{k(k+1)}{\lambda^2} - \frac{k^2}{\lambda^2}$$

$$= \frac{k^2 + k - k^2}{\lambda^2}$$

$$\text{Var}(X) = \frac{k}{\lambda^2}.$$

Moments and hence Mean and Variance

By the definition of r^{th} moments about origin,

$$\mu_r{}' = E\left[x^r\right]$$

$$= \int_0^\infty x^r f(x)\, dx$$

$$= \int_0^\infty x^r \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{(k+r)-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \frac{\Gamma(k+r)}{\lambda^{k+r}} \quad \left(\because \int_0^\infty x^{n-1} e^{-ax} dx = \frac{\Gamma(n)}{a^n} \right)$$

$$= \frac{\Gamma(k+r)}{\Gamma(k)} \frac{1}{\lambda^r}$$

$$\mu_r{}' = \frac{1}{\lambda^r} \frac{\Gamma(k+r)}{\Gamma(k)} \qquad (1)$$

To find Mean and Variance

Put r=1 in (1), we get

$$\mu_1{}' = \frac{1}{\lambda} \frac{\Gamma(k+r)}{\Gamma(k)}$$

$$= \frac{1}{\lambda} \frac{k!}{(k-1)!}$$

$$= \frac{1}{\lambda} \frac{k(k-1)!}{(k-1)!}$$

$$\mu_1' = \frac{k}{\lambda}.$$

Put r=2 in (1), we get

$$\mu_2' = \frac{1}{\lambda^2}\frac{\Gamma(k+2)}{\Gamma(k)}$$

$$= \frac{1}{\lambda^2}\frac{(k+1)!}{(k-1)!}$$

$$= \frac{1}{\lambda^2}\frac{(k+1)k(k-1)!}{(k-1)!}$$

$$\mu_2' = \frac{k(k+1)}{\lambda^2}$$

$$\operatorname{Var}(X) = \mu_2$$

$$= \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{k(k+1)}{\lambda^2} - \frac{k^2}{\lambda^2}$$

$$= \frac{k^2 + k - k^2}{\lambda^2}$$

$$\operatorname{Var}(X) = \frac{k}{\lambda^2}.$$

Additive Property (or) Productive Property

Statement

The sum of a finite number of independent Erlang variables is also a Erlong variable, that isif $X_1, X_2, ..., X_n$ are independent Erlang variables with parameters $(\lambda, k_1), (\lambda, k_2), ..., (\lambda, k_n)$, then $X_1 + X_2 + ... + X_n$ is also an Erlang variable with parameter $(\lambda, k_1 + k_2 + ... + k_n)$.

Proof

The mgf of Erlang distribution is $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-k}$

We know that,

$$M_{X1+X2+\ldots+Xn}(t)=M_{X1}(t)\ M_{X2}(t)\ \ldots\ M_{Xn}(t)$$

$$=\left(1-\frac{t}{\lambda}\right)^{-k_1}\left(1-\frac{t}{\lambda}\right)^{-k_2}\ldots\left(1-\frac{t}{\lambda}\right)^{-(k_1+k_2+\ldots+k_n)}$$

$$=\left(1-\frac{t}{\lambda}\right)^{-(k_1+k_2+\ldots+k_n)}$$ which is the mgf of Erlang distribution with parameters $(\lambda, k_1+k_2+\ldots+k_n)$.

$\therefore$ The sum of n independent Erlang variables is also an Erlang variable.

Simple Gamma Distribution (or) Simple Erlang Distribution (One Parameter Form)

When λ=1 the general Gamma distribution (2 parameters) form is called one parameter form or simple Gamma Distribution.

Definition

A continuous random variable X is said to follow simple Gamma distribution with parameter k, if its pdf is given by,

$$f(x)=\begin{cases}\dfrac{1}{\Gamma(k)}x^{k-1}e^{-x} & x\geq 0, k>0\\ 0 & otherwise\end{cases}$$

Note

When k=1 the general Gamma distribution reduces to exponential distribution $f(x)=\lambda e^{-\lambda x}$ with parameter λ.

MGF and hence Mean and Variance of Simple Gamma Distribution

By the definition of mgf,

$$M_X(t)=E\left[e^{tx}\right]$$

$$=\int_0^{\infty} e^{tx}\frac{1}{\Gamma(k)}x^{k-1}e^{-x}\,dx$$

$$= \frac{1}{\Gamma(k)} \int_0^\infty x^{k-1} e^{tx-x} dx$$

$$= \frac{1}{\Gamma(k)} \int_0^\infty x^{k-1} e^{-(1-t)x} dx$$

$$= \frac{1}{\Gamma(\mathrm{k})} \frac{\Gamma(\mathrm{k})}{(1-\mathrm{t})^{\mathrm{k}}}$$

$$= \frac{1}{(1-\mathrm{t})^{\mathrm{k}}}$$

$$M_X(t) = (1-t)^{-k}$$

To find Mean and Variance

$$\mathrm{M_X(t)} = (1-\mathrm{t})^{-\mathrm{k}}$$

We know that,

$$(1-\mathrm{x})^{-\mathrm{n}} = 1 + \frac{\mathrm{n}}{1!}\mathrm{x} + \frac{\mathrm{n(n+1)}}{2!}\mathrm{x}^2 + \ldots$$

$$(1-\mathrm{t})^{-\mathrm{k}} = 1 + \frac{\mathrm{k}}{1!}\mathrm{t} + \frac{\mathrm{k(k+1)}}{2!}\mathrm{t}^2 + \ldots$$

$$= 1 + \frac{\mathrm{t}}{1!}\mathrm{k} + \frac{\mathrm{t}^2}{2!}\mathrm{k(k+1)} + \ldots$$

$\therefore$ The Coefficient of $\frac{t}{1!}$ $is\ k = \mu_1{}'$, $\therefore \mu_1{}' = k = Mean$.

The Coefficient of $\frac{t^2}{2!}$ $is\ k(k+1) = \mu_2{}'$

$$\therefore \mu_2 = Var(X) = \mu_2{}' - \left(\mu_1{}'\right)^2$$

$$= \mathrm{k(k+1)} - \mathrm{k}^2$$

$$= \mathrm{k}^2 + \mathrm{k} - \mathrm{k}^2$$

$$\text{Var}(x) = k$$

Relationship between a Gamma variate and normal variate

Let X be a normal variate with parameters μ and σ^2. Then the variate Y given by the transformation $Y = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ becomes a Gamma variate with parameter $\frac{1}{2}$.

Proof

Given X be a normal variate with parameter μ and σ^2. Then its pdf is,

$$f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad -\infty < x < \infty$$

$$-\infty < \mu < \infty, \ \sigma^2 > 0$$

$$\text{Let, } y = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \text{ then } \Rightarrow \left(\frac{x-\mu}{\sigma}\right)^2 = 2y$$

$$\Rightarrow \frac{x-\mu}{\sigma} = \sqrt{2y}$$

$$\therefore \frac{dy}{dx} = \left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}$$

$$\Rightarrow dy = \frac{\sqrt{2y}}{\sigma} dx$$

$$\Rightarrow dx = \frac{\sigma}{\sqrt{2y}} dy$$

$\therefore$ The pdf of y is

$$f(y)dy = \frac{1}{\sigma\sqrt{2\pi}} e^{-y} . \frac{\sigma}{\sqrt{2y}} dy \ -\infty < y < \infty$$

$$f(y)dy = \frac{1}{\Gamma\left(\frac{1}{2}\right)} e^{-y} y^{-\frac{1}{2}} dy, \ 0 < y < \infty$$

Which is the pdf of simple Gamma distribution with parameter $\frac{1}{2}$.

Theorem 1

Show that under certain conditions simple Gamma distribution tends to normal distribution. In other words, show that the limiting form of Gamma distribution is normal.

Proof

Let X be a Gamma variate with parameter k, then its pdf is,

$$f(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}, \quad 0 \le x < \infty, k > 0$$

Mean and Variance of the Gamma variate are equal and given by k.

$\therefore$ Mean = V(X) =k.

Then the standard gamma variate is defined as $z = \frac{X-k}{\sqrt{k}}$.

MGF of z is,

$$M_z(t) = E\left[e^{tz}\right]$$

$$= E\left[e^{t\left(\frac{X-k}{\sqrt{k}}\right)}\right]$$

$$= E\left[e^{\frac{tX}{\sqrt{k}}} e^{\frac{-tk}{\sqrt{k}}}\right]$$

$$= E\left[e^{\frac{tX}{\sqrt{k}}} e^{-t\sqrt{k}}\right]$$

$$= e^{-t\sqrt{k}} E\left[e^{\frac{tx}{\sqrt{k}}}\right]$$

$$= e^{-t\sqrt{k}} \left(1 - \frac{t}{\sqrt{k}}\right)^{-k} \quad \left(\because M_X(t) = (1-t)^{-k}\right)$$

Taking logarithms on both sides,

$$\log M_z(t)=\log\left(e^{-t\sqrt{k}}\left(1-\frac{t}{\sqrt{k}}\right)^{-k}\right)$$

$$=-t\sqrt{k}-k\log\left(1-\frac{t}{\sqrt{k}}\right)$$

$$=-t\sqrt{k}+k\left(-\log\left(1-\frac{t}{\sqrt{k}}\right)\right)$$

$$=-t\sqrt{k}+k\left[\frac{t}{\sqrt{k}}+\frac{t^2}{2k}+\frac{t^3}{3k\sqrt{k}}+\ldots\right]\left(\because -\log(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)$$

$$=-t\sqrt{k}+\frac{tk}{\sqrt{k}}+\frac{t^2}{2}+\frac{t^3}{3\sqrt{k}}+\ldots$$

Taking limit $k\to\infty$ on both sides we get,

$$\lim_{k\to\infty}\log M_z(t)=\frac{t^2}{2}$$

$$\lim_{k\to\infty}e^{\log M_Z(t)}=e^{\frac{t^2}{2}}$$

$$\lim_{k\to\infty}M_Z(t)=e^{\frac{t^2}{2}}$$

which is the mgf of standard normal variate. Therefore, limiting case of standard Gamma variate becomes normal variate.

4.4 Beta Distribution of first and second kinds

Beta distribution of first kind

Definition

A continuous r.v X is said to follow a beta variate of first kind if its pdf is given by,

$$f(x)=\frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)},\ 0<x<1,\ m>0\ and\ n>0$$

This distribution is called beta distribution of first kind with parameters m and n.

Since the total probability is equal to 1, we have $\int_0^1 f(x)\,dx = 1$

$$\Rightarrow \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)}\,dx = 1$$

$$\Rightarrow \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

$$\therefore \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Note

The MGF for the beta distribution does not have a simple form. However the moments are readily found by using their definition.

Moments of Beta distribution of first kind

By the definition of r^{th} moment about origin is given by,

$$\mu_r' = E[X^r]$$

$$= \int_0^1 x^r f(x)\,dx$$

$$= \int_0^1 x^r \frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)}\,dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{m+r-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{\beta(m,n)}\,\beta(m+r,n)$$

$$\mu_r' = \frac{\beta(m+r,n)}{\beta(m,n)}$$

We know that

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \mu_r' = \frac{\Gamma(m+r)\Gamma(n)}{\Gamma(m+n+r)} \Big/ \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\Gamma(m+r)\Gamma(n)}{\Gamma(m+n+r)} \times \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}$$

$$\mu_r' = \frac{\Gamma(m+r)\Gamma(m+n)}{\Gamma(m+n+r)\Gamma(m)} \qquad (1)$$

which is the r^{th} moment about origin.

To find Mean and Variance

Put r=1 in (1), we get

$$\mu_1' = \frac{\Gamma(m+1)\Gamma(m+n)}{\Gamma(m+n+1)\ \ \Gamma(m)}$$

$$= \frac{m\Gamma(m)\ \Gamma(m+n)}{m+n\Gamma(m)+n\ \Gamma(m)} \qquad (\because \Gamma(n+1)=n\Gamma(n))$$

$$\mu_1' = \frac{m}{m+n}$$

Put r=2 in (1), we get

$$\mu_2' = \frac{\Gamma(m+2)\Gamma(m+n)}{\Gamma(m+n+2)\ \ \Gamma(m)}$$

$$= \frac{(m+1)\,m\Gamma(m)\ \ \Gamma(m+n)}{(m+n+1)(m+n)\Gamma(m+n)\Gamma(m)}$$

$$\mu_2' = \frac{m(m+1)}{(m+n)(m+n+1)}$$

$$\therefore \mu_2 = Var(X) = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m^2}{(m+n)^2}$$

$$=\frac{m}{(m+n)}\left[\frac{m+1}{m+n+1}-\frac{m}{m+n}\right]$$

$$=\frac{m}{(m+n)}\left[\frac{(m+1)(m+n)-m(m+n+1)}{(m+n+1)(m+n)}\right]$$

$$=\frac{m}{(m+n)}\ \frac{n}{(m+n+1)(m+n)}$$

$$\mu_2=\frac{mn}{(m+n+1)(m+n)^2}\ .$$

Beta distribution of second kind

Definition

A continuous r.v X is said to be beta variable of second kind if its pdf is given by,

$$f(x)=\frac{1}{\beta(m,n)}\frac{x^{m-1}}{(1-x)^{m+n}},\quad 0<x<\infty,\ m>0\,and\,n>0$$

This distribution is called beta distribution of second kind with parameter m and n.

Since the total probability is 1, we have

$$\int_0^\infty f(x)\,dx=1$$

$$\Rightarrow\int_0^\infty\frac{1}{\beta(m,n)}\frac{x^{m-1}}{(1-x)^{m+n}}\,dx=1$$

$$\Rightarrow\frac{1}{\beta(m,n)}\int_0^\infty\frac{x^{m-1}}{(1-x)^{m+n}}\,dx=1$$

$$\Rightarrow\int_0^\infty\frac{x^{m-1}}{(1-x)^{m+n}}\,dx=\beta(m,n)$$

$$\therefore\beta(m,n)=\int_0^\infty\frac{x^{m-1}}{(1-x)^{m+n}}\,dx$$

Moments of beta distribution of second kind

By the definition of r^{th} moment about origin,

$$\mu_r' = E[X^r]$$

$$= \int_0^\infty x^r \ f(x)\, dx$$

$$= \int_0^\infty x^r \ \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1-x)^{m+n}}\, dx$$

$$= \frac{1}{\beta(m,n)} \int_0^\infty \frac{x^{m+r-1}}{(1-x)^{m+n}}\, dx$$

$$= \frac{1}{\beta(m,n)} \int_0^\infty \frac{x^{(m+r)-1}}{(1+x)^{m+r+n-r}}$$

$$= \frac{1}{\beta(m,n)} \beta(m+r, n-r)$$

$$= \frac{\beta(m+r, n-r)}{\beta(m,n)}$$

$$= \frac{\Gamma(m+r)\ \Gamma(n-r)}{\Gamma(m+r+n-r)} \Big/ \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\Gamma(m+r)\ \Gamma(n-r)}{\Gamma(m+r+n-r)} \times \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}$$

$$\mu_r' = \frac{\Gamma m+r\ \Gamma n-r}{\Gamma(m)\ \Gamma(n)} \qquad (1)$$

To find Mean and Variance

Put r=1 in equation (1) we get,

$$\mu_1' = \frac{\Gamma(m+1)\ \Gamma(n-1)}{\Gamma(m)\ \Gamma(n)}$$

$$= \frac{m\Gamma(m)\ \Gamma(n-1)}{\Gamma(mn-1)\Gamma(n-1)} \qquad (\because \Gamma n = (n-1)\Gamma(n-1))$$

$$\mu_1' = \frac{m}{n-1} = \text{Mean}$$

Put r=2 in equation (1) we get,

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{(m+1)m}{(n-1)(n-2)} - \frac{m^2}{(n-1)^2}$$

$$= \frac{m}{n-1}\left[\frac{m+1}{n-2} - \frac{m}{n-1}\right]$$

$$= \frac{m}{n-1}\left[\frac{(m+1)(n-1) - m(n-2)}{(n-1)(n-2)}\right]$$

$$= \frac{m}{n-1}\left[\frac{mn - m + n - 1 - mn + 2m}{(n-1)(n-2)}\right]$$

$$\mu_2 = \frac{m(m+n-1)}{(n-1)^2(n-2)}$$

Problem 1:

Let X and Y be the two independent Gamma variate with parameter m and n respectively. Then the variates U=X+Y and $V = \frac{X}{X+Y}$ for independent and the variable U is the Gamma variate with parameter m+n and V is the β variate of first kind with parameter m and n.

Solution:

Given X and Y be the two independent gamma variates with parameter m and n. Then their probability density functions are,

$$f(x)\,dx = \frac{x^{m-1}\,e^{-x}}{\Gamma(m)}\,dx; \quad 0 \le x < \infty,\ m > 0$$

and

$$f(y)\,dy = \frac{y^{n-1}\,e^{-y}}{\Gamma(n)}\,dy; \quad 0 \le y < \infty,\ n > 0$$

Also given X and Y are independent, the joint probability function can be written as,

$$f(x,y)\,dx\,dy = f_x(x)\,f_y(y)\,dx\,dy$$

$$= \frac{x^{m-1}e^{-x}}{\Gamma m}\;\frac{y^{m-1}e^{-y}}{\Gamma n}\,dx\,dy$$

$$\therefore f(x,y)\,dx\,dy = \frac{1}{\Gamma m\,\Gamma n}\,e^{-(x+y)}\,x^{m-1}y^{n-1}\,dx\,dy \qquad (1)$$

Let,

$$U = X + Y \quad \text{and} \quad V = \frac{X}{X+Y}$$

$$\Rightarrow Y = U - X \qquad \Rightarrow V = \frac{X}{U} \Rightarrow X = UV$$

$$= U - UV$$

$$Y = U(1-V)$$

$$X = UV; \quad Y = U(1-V)$$

For this transformation, the product dxdy in equation (1) is modified as |J|.dudv.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ (1-v) & -u \end{vmatrix}$$

$$= -uv - u(1-v)$$

$$= -uv - u + uv$$

$$J = -u$$

$$\therefore |J| = u$$

Given $0 \le x \le \infty$, $0 \le y \le \infty$.

When X=0, UV=0

$\Rightarrow$ u=0 or v=0.

When Y=0, u(1-v) =0

$\Rightarrow$ u=0 or v=1

When x=∞, uv=∞

$\Rightarrow$ u=∞ or v=∞.

$\Rightarrow$ u=0 to ∞ and v=0 to 1.

The joint density function of U and V is from equation (1) is,

$$f(u,v)\,dudv = \frac{1}{\Gamma(m)\Gamma(n)} e^{-u} \left(uv\right)^{m-1} \left[u(1-v)\right]^{n-1} |J| dudv$$

$$= \frac{1}{\Gamma(m)\Gamma(n)} e^{-u} u^{m-1+n-1} v^{m-1} (1-v)^{n-1} u\, dudv$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n\Gamma m+n)} e^{-u}\ u^{m+n-1} v^{m-1} (1-v)^{n-1}\ dudv$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} e^{-u}\ \frac{u^{m+n-1}\ du\, v^{m-1}}{\Gamma(m+n)} (1-v)^{n-1}\ dv$$

$$= \frac{1}{\Gamma(m+n)} e^{-u}\ u^{m+n-1}\ du\ \frac{1}{\beta(m,n)} v^{m-1} \left(1-v\right)^{n-1}\ dv$$

$$\mathrm{f(u,v)\,dudv = f(u) f(v)\,dudv}$$

$\therefore$ U and V are independent. Also U is gamma variate with parameter (m+n) and V is a beta distributions of first kind.

Exercise

1. Derive the moments of beta distribution of first kind.
2. Derive the moments of Gamma distribution and hence find its mean and variance.
3. Define exponential distribution and state its constants.
4. Establish the relationship between a gamma variate and normal variate.
5. Derive the median of exponential distribution.
6. Derive the mean and variance of gamma distribution and comment on its additive property.
7. Derive the r^{th} raw moments of beta distribution of second kind and find its mean and variance.

8. Let X and Y be two independent gamma variate with parameter m and n respective. Then prove that the variate U = X+Y, $V = \frac{X}{Y}$ are independent and U is a gamma variate with parameter m+n and V is a beta variate of second kind with parameters m and n.

Unit – V

Sampling Distributions

5.1 Introduction

5.2 Chi-square Distribution

5.3 Student's t Distribution

5.4 F-Distribution

5.1 Introduction

The entire large sample theory was based on the application of Normal test. However, if the sample size n is very small, the distribution of the various statistics, e.g., $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$ or $Z = (X - nP)/\sqrt{Npq}$ etc., are far from normality and as such normal test can not be applied if n is small. In such cases exact sample tests, pioneered by Gosset (1908) who wrote under the pen name of Student, and later on developed and extended by R.A.Fisher (1926), are used. The exact sample tests can be applied to large samples. In all the exact sample tests, the basic assumption is that the populations from which samples are drawn are normal, that is parent populations are normally distributed.

5.2 Chi-square distribution

The square of a standard normal variance is known as χ^2 – variate with one degrees of freedom. In general, if X_1, X_2, …, X_n are n independent normal variables with mean μ_i and variables σ_i^2, then

$\chi^2 = \sum_{i=1}^{n}\left(\frac{x_i - \mu_i}{\sigma_i}\right)$ is a χ^2 variate with n degrees of freedom.Thus, if X ~ N(μ, σ^2) then $z = \frac{x-\mu}{\sigma} \sim N(0,1)$.

$\therefore z^2 = \left(\frac{x-\mu}{\sigma}\right)^2$ is a χ^2 variate with 1 degrees of freedom.

Definition

A r.v X is said to have a χ^2 distribution with n degrees of freedom if its pdf is given by

$$f\left(\chi^2\right) = \frac{x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)};\ 0 \le x < \infty.$$

Note:

When $\alpha = \frac{n}{2}$ and λ=2,the gamma distribution becomes χ^2 distribution.

MGF, Mean and Variance

By the definition of MGF,

$$M_X(t) = E\left[e^{tx}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_{0}^{\infty} \frac{x^{\frac{n}{2}} e^{-\frac{x}{2}} e^{tx}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}dx$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_{0}^{\infty} x^{\frac{n}{2}-1} e^{-\left(\frac{1}{2}-t\right)x} dx$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\left(\frac{1}{2}-t\right)^{\frac{n}{2}}}$$

$$= \frac{1}{2^{\frac{n}{2}}} \frac{1}{\left(\frac{1-2t}{2}\right)^{\frac{n}{2}}}$$

$$= \frac{1}{2^{\frac{n}{2}}} \frac{2^{\frac{n}{2}}}{(1-2t)^{\frac{n}{2}}}$$

$$M_X(t) = (1-2t)^{-\frac{n}{2}}$$

$$\mathrm{M_X(t)} = (1-2\mathrm{t})^{-\frac{n}{2}}$$

$$= 1 + \frac{n}{2}2t + \frac{n}{2}\left(\frac{n}{2}+1\right) \Big/ 2!\,(2t)^2 + \ldots$$

$$= 1 + \mathrm{n}\frac{\mathrm{t}}{1!} + 4\frac{\mathrm{n}}{2}\left(\frac{\mathrm{n}}{2}+1\right)\frac{\mathrm{t}^2}{2!} + \ldots$$

The Coefficient of $\frac{t}{1!} is\, n = \mu_1' = Mean$

Mean = n

The Coefficient of $\frac{t^2}{2!} is \frac{4n}{2}\left(\frac{n}{2}+1\right)$

$$= n^2 + 2n$$
$$= \mu_2'$$

$$\mathrm{Var(X)} = \mu_2 = \mu_2' - \left(\mu_1'\right)^2$$

$$= \mathrm{n}^2 + 2\mathrm{n} - \mathrm{n}^2$$

$$\mathrm{Var}(X) = 2n$$

Moments, Mean and Variance

By the definition of r^{th} moment about origin,

$$\mu_r{}' = E\left[X^r\right]$$

$$= \int_{-\infty}^{\infty} x^r\, f(x)\, dx$$

$$= \int_0^{\infty} x^r \frac{1}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}\, dx$$

$$= \int_0^{\infty} \frac{1}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)} x^{\left(\frac{n}{2}+r\right)-1} e^{-\frac{x}{2}}\, dx$$

$$= \frac{1}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{2}+r\right)}{\left(\frac{1}{2}\right)^{\frac{n}{2}+r}}$$

$$= \frac{1}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)} 2^{\frac{n}{2}}\, 2^r\, \Gamma\left(\frac{n}{2}+r\right)$$

$$= \frac{1}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)} 2^{\frac{n}{2}}\, 2^r\, \Gamma\left(\frac{n}{2}+r\right)$$

$$\mu_r{}' = \frac{2^r\, \Gamma\left(\frac{n}{2}\right)+r}{\Gamma\left(\frac{n}{2}\right)}$$

Put r=1,

$$\mu_1' = \frac{2\Gamma\left(\frac{n}{2}\right)+1}{\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{2\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$\mu_1' = n$$

Put r=2,

$$\mu_2' = \frac{2^2\Gamma\left(\frac{n}{2}\right)+2}{\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{2^2\frac{n}{2}\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$\mu_2' = n^2 + 2n$$

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2$$

$$= n^2 + 2n - n^2$$

$$\mu_2 = 2n = \mathrm{Var}(X)$$

Cummulants of χ^2 distribution

By definition of cummulants

$$k_X(t) = \log M_X(t)$$

$$= \log(1-2t)^{-\frac{n}{2}}$$

$$= -\frac{n}{2}\log(1-2t)$$

$$= \frac{n}{2}\left(-\log(1-2t)\right)$$

$$= \frac{n}{2}\left[2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{3} + \frac{(2t)^4}{4} + \dots\right]$$

$$= \frac{n}{2}\left[2\frac{t}{1!} + 4\frac{t^2}{2!} + 3!\frac{8}{3}\frac{t^3}{3!} + \frac{16}{4}4!\frac{t^4}{4!} + \dots\right]$$

$$k_1 = \text{The Coefficient of } \frac{t^1}{1!} \text{ in } k_x(t) = \frac{n}{2}.2$$

Mean = μ_1 = n

$$k_2 = \text{The Coefficient of } \frac{t^2}{2!} \text{ in } k_x(t) = \frac{n}{2}.4 = 2n = \text{var} = \mu_2$$

$$k_3 = \text{The Coefficient of } \frac{t^3}{3!} \text{ in } k_x(t) = \frac{n}{2}.16 = 8n = \mu_3$$

$$k_4 = \text{The Coefficient of } \frac{t^4}{4!} \text{ in } k_x(t) = \frac{n}{2}.\frac{16}{4} \times 4! = 48n = \mu_4$$

Hence k_1, k_2, k_3, k_4 are first four central moments.

Limiting form of χ^2 – distribution

Consider a χ^2 – variate with n – degrees of freedom. It is known that the mean and variance of χ^2 – variate respectively given by $E(X^2) = n$ and $V(X^2) = 2n$

Let us define the standard χ^2 – variate is $z = \frac{x^2 - \mu}{\sigma} = \frac{x^2 - n}{\sqrt{2n}}$

$$M_z(t) = E\left[e^{tz}\right]$$

$$= E\left[e^{t\left(\frac{x^2-n}{\sqrt{2n}}\right)}\right]$$

$$= E\left[e^{\frac{tx^2}{\sqrt{2n}}} e^{-\frac{t\,n}{\sqrt{2n}}}\right]$$

$$M_z(t) = e^{-t\sqrt{\frac{n}{2}}} . \left(1 - \frac{2t}{\sqrt{2n}}\right)^{-\frac{n}{2}}$$

$$\log M_z(t) = -t\sqrt{\frac{n}{2}} + \frac{n}{2}\left(-\log\left(1 - \frac{2t}{\sqrt{2n}}\right)\right)$$

$$\log M_z(t) = -t\sqrt{\frac{n}{2}} + \frac{n}{2}\left\{\frac{2t}{\sqrt{2n}} + \frac{\left(\frac{2t}{\sqrt{2n}}\right)^2}{2} + \frac{\left(\frac{2t}{\sqrt{2n}}\right)^3}{3} + ...\right\}$$

$$= -t\sqrt{\frac{n}{2}} + t\sqrt{\frac{n}{2}} + \frac{n}{2}\left(\frac{2t}{\sqrt{2n}}\right)^2 \Big/ 2 + \frac{n}{2}\left(\frac{2t}{\sqrt{2n}}\right)^3 \Big/ 3 + ...$$

$$= \frac{n}{2} . \frac{\frac{4t^2}{2n}}{2} + \frac{n}{2} . \frac{\left(\frac{2t}{\sqrt{2n}}\right)^2}{2} + \ldots$$

$$= \frac{t^2}{2} + \frac{n}{2}\left(\frac{2t}{\sqrt{2n}}\right)^3 \Big/ 3 + ...$$

Take as $\lim_{n \to \infty}$

$$\lim_{n\to\infty} \log M_z(t) = \frac{t^2}{2}.$$

$\Rightarrow \lim_{n\to\infty} M_z(t) = e^{\frac{t^2}{2}}$ which is MGF of Standard normal variate. Hence by uniqueness theorem, the variable z is a standard normal variate.So the limiting form of χ^2 distribution is normal distribution.

Applications of χ^2 distribution

(i) It is used to test if the hypothetical values of population variance is $\sigma^2 = \sigma_0^2$.

(ii) It is used to test the goodness of fit.

(iii) It is used to test the independence of attributes.

(iv) It is used to test the homogeneity of independence estimates of the population correlation coefficient.

5.3 Student's t – distribution

Definition

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a normal population with mean μ and σ^2. Then the t-statistic is defined by,

$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$ *where* $\bar{x} = \frac{1}{n}\Sigma x_i$, x_i is sample mean and $S^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x})$, S^2 is an unbiased estimator of the population variance σ^2 and it follows Student's t distribution with γ = n-1 degrees of freedom with pdf is,

$$f(t) = \frac{1}{\sqrt{\gamma}\,\beta\left(\frac{1}{2}, \frac{\gamma}{2}\right)} \frac{1}{\left(1 + \frac{t^2}{\gamma}\right)^{\frac{\gamma+1}{2}}}$$

Derivation of Student's t – distribution

Consider a normal population with mean μ and variance σ^2,where σ^2 is unknown.

Let x_1, x_2, …,x_n be a random sample of size n from this normal population. Then the student t statistic is defined by,

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} \qquad where\, \bar{x} = \frac{1}{n}\Sigma x_i$$

$$S^2 = \frac{1}{n-1}\Sigma(x - \bar{x})^2$$

$$\Rightarrow (n-1)S^2 = \Sigma(x_i - \bar{x})^2$$

$$\therefore t^2 = \frac{(\bar{x} - \mu)^2}{S^2/n}$$

We know that, the sample variance,

$$s^2 = \frac{1}{n}\Sigma(x_i - \bar{x})^2$$

$$ns^2 = \Sigma(x_i - \bar{x})^2 = (n-1)S^2$$

$$\Rightarrow S^2 = \frac{n}{1-n}s^2$$

$$t^2 = \frac{((\bar{x} - \mu)^2}{\frac{n}{n-1}\frac{s^2}{n}}$$

$$= \frac{n(n-1)(\bar{x} - \mu)^2}{ns^2}$$

$$\frac{t^2}{n-1} = \frac{n(\bar{x} - \mu)^2}{ns^2}$$

$$= \frac{n(\bar{x} - \mu^2/\sigma^2)}{ns^2/\sigma^2}$$

$$\Rightarrow \frac{t^2}{\gamma} = \left(\frac{\bar{x} - \mu}{S/\sqrt{n}}\right)^2 . \frac{1}{ns^2/\sigma^2} \qquad (1)$$

We know that if x_i follows N(μ, σ^2),then $\bar{x} \sim N(\mu, \sigma/\sqrt{n})$.

If $\bar{x} \sim N(\mu, \sigma/\sqrt{n})$. Then $\left(\frac{\bar{x} - \mu}{S/\sqrt{n}}\right) \sim N(0,1)$

$$\therefore \left(\frac{\bar{x}-\mu}{S/\sqrt{n}} \right)^2 \sim \chi^2 \text{ with 1 degrees of freedom.}$$

Also we know that, the sample variance

$$s^2 = \frac{1}{n}\Sigma(x_i - \bar{x})^2$$

$$\frac{ns^2}{\sigma^2} = \frac{\Sigma(x_i - \bar{x})^2}{\sigma^2}$$

$$= \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma} \right)^2 \sim \chi^2 \text{ with (n-1) degrees of freedom.}$$

$\therefore \frac{t^2}{\gamma}$ in (1) is a ratio of two independent χ^2 variates with one and (n-1)degrees of freedom.

Also we known that, the ratio of two independent χ^2 variates is a β – variate of second kind with parameter $\frac{1}{2}$ and $\frac{n-1}{2}$.

$\therefore$ The pdf of t can be written as,

$$f(x) = \frac{1}{\beta(m,n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}}, \; 0 < x < \infty, m > 0 \; \& \; n > 0$$

Here $x = \frac{t^2}{\gamma}$, $\quad \therefore dx = \frac{2t}{\gamma} dt$

$$\therefore f(t)dt = \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \cdot \frac{\left(\frac{t^2}{\gamma}\right)^{\frac{1}{2}-1}}{\left(1+\frac{t^2}{\gamma}\right)^{\frac{1}{2}+\frac{n-1}{2}}} \frac{2t}{\gamma} dt \qquad for \; 0 \le t^2 \le \infty$$

After simplification, we get,

$$f(t)dt = \frac{1}{\sqrt{\gamma}\beta\left(\frac{1}{2},\frac{\gamma}{2}\right)} \frac{1}{\left(1+\frac{t^2}{\gamma}\right)^{\frac{r+1}{2}}} dt \qquad for -\infty < t < \infty$$

Which is the pdf of t-distribution with γ = n-1 degrees of freedom.

Limiting case of t – distribution

The density function of t – distribution with n degrees of freedom is,

$$f(t) = \frac{1}{\sqrt{n}\beta\left(\frac{1}{2},\frac{n}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}; \qquad -\infty < t < \infty$$

As $n \to \infty$ we have

$$\lim_{n\to\infty} \quad f(t) = \lim_{n\to\infty} \frac{1}{\sqrt{n}\beta\left(\frac{1}{2},\frac{n}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$$= \left(\lim_{n\to\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{n}{2}\right)} \right) \left(\lim_{n\to\infty} \left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}} \right) \qquad (1)$$

By Sterling formula,

$$\lim_{R\to\infty} \frac{\Gamma(n+k)}{\Gamma n} = n^k$$

$$= \frac{1}{\Gamma(1/2)}\left(\frac{n}{2}\right)^{\frac{1}{2}} \lim_{n\to\infty}\left\{1-\frac{n}{2}\frac{t^2}{n}+\frac{\left(\frac{-n}{2}\right)\left(\frac{-n}{2}-1\right)}{2!}\left(\frac{t^2}{n}\right)^2+\ldots\right\}\frac{1}{\sqrt{n}}$$

After simplification and applying limits, we get

$$f(t)=\frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}$$

which is the pdf of standard normal distribution. Therefore limiting form of t-distribution is normal.

Constants of t-distribution

Prove that for t – distribution, all odd order moments vanished and even order moments exists. That is,

i) $\mu_{2r+1} = 0 \quad for \;\; r = 0,1,2,..$

ii) $\mu_{2r} = \dfrac{n^r\,(2r-1)(2r-2)...3.1}{(n-2)(n-4)...(n-r)}$

Proof:

Since f(t) is symmetrical about the line t=0, all the moments of odd order about origin vanish (i.e.) $\mu_{2r+1}{}' = 0 \quad for \;\; r = 0,1,2,..$

In particular, $\mu_1{}' = 0 = Mean$. Hence the central moments coincide with moments about origin.

$$\therefore \mu_{2r+1} = 0 \quad for \quad r = 0,1,2,..$$

The moments of even order are given by,

$$\mu_{2r} = \mu_{2r}{}' = E(t)^{2r}$$

$$= \int_{-\infty}^{\infty} t^{2r} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} t^{2r} \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt$$

$$= \frac{2}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_{0}^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt$$

Put $y = \dfrac{1}{1+\frac{t^2}{n}}$

$$\Rightarrow \frac{1}{y} = 1 + \frac{t^2}{n} \Rightarrow \frac{t^2}{n} = \frac{1}{y} - 1$$

$$\Rightarrow t^2 = n\left(\frac{1}{y} - 1\right) = \frac{n(1-y)}{y}$$

$$t = \sqrt{n}\left(\frac{1}{y} - 1\right)^{\frac{1}{2}}$$

$$dt = \sqrt{n}\,\frac{1}{2}\left(\frac{1}{y}-1\right)^{\frac{1}{2}-1}\left(-\frac{1}{y^2}\right)dy$$

When t=0, y=1; when t=∞, y=0

$$\therefore \mu_{2r} = \frac{2}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)}\int_1^0 \left(\frac{n(1-y)}{y}\right)^r y^{\frac{n+1}{2}} \left(-\sqrt{n}\,\frac{1}{2}\left(\frac{1-y}{y}\right)^{-\frac{1}{2}}\frac{1}{y^2}\right)dy$$

$$= \frac{n^r}{\beta\left(\frac{1}{2},\frac{n}{2}\right)}\int_0^1 (1-y)^{r-\frac{1}{2}}\, y^{\frac{n+1}{2}-r+\frac{1}{2}-2}\,dy$$

$$= \frac{n^r}{\beta\left(\frac{1}{2},\frac{n}{2}\right)}\int_0^1 y^{\frac{n}{2}-r-1}\,(1-y)^{r+\frac{1}{2}-1}\,dy$$

By beta distribution of first kind, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$\therefore \mu_{2r} = \frac{n^r}{\beta\left(\frac{1}{2},\frac{n}{2}\right)}\beta\left(\frac{n}{2}-r, r+\frac{1}{2}\right)$$

$$= \frac{n^r}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)\Big/\Gamma\left(\frac{1}{2}\right)+\left(\frac{n}{2}\right)} \times \frac{\Gamma\left(\frac{n}{2}\right)-r\,\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}-r+r+\frac{1}{2}\right)}$$

$$= \frac{n^r}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \times \Gamma\left(\frac{1}{2}+\frac{n}{2}\right) \times \frac{\Gamma\left(\frac{n}{2}-r\right)\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}$$

$$= \frac{n^r \, \Gamma\left(\frac{n}{2} - r\right) \Gamma\left(r + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)}$$

$$= n^r \, \Gamma\left(\frac{n}{2} - r\right) \frac{\left(r + \frac{1}{2} - 1\right)\left(r + \frac{1}{2} - 2\right) \ldots \frac{3}{2} \frac{1}{2} \Gamma \frac{1}{2}}{\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right) \ldots \left(\frac{n}{2} - r\right) \Gamma\left(\frac{n}{2} - r\right)}$$

$$= \frac{\frac{1}{2^r} n^r \, (2r-1)(2r-3)\ldots 3.1}{\frac{1}{2^r} \, (n-2)(n-4)\ldots(n-2r)}$$

$$\mu_{2r} = \frac{n^r \, (2r-1)(2r-3)\ldots 3.1}{(n-2)(n-4)\ldots(n-2r)}$$

In particular, put r=1,

$$\mu_2 = \mathrm{n} \frac{1}{\mathrm{n}-2} = \frac{\mathrm{n}}{\mathrm{n}-2}$$

$$\mathrm{Var}(\mathrm{X}) = \frac{\mathrm{n}}{\mathrm{n}-2}$$

Put r=2,

$$\mu_4 = \frac{\mathrm{n}^2 \, 3.1}{(\mathrm{n}-2)(\mathrm{n}-4)}$$

$$\mu_4 = \frac{3\mathrm{n}^2}{(\mathrm{n}-2)(\mathrm{n}-4)}$$

And so on.

$$\therefore \mu_1 = Mean = 0 \; and \; \mu_2 = Var(X) = \frac{n}{n-2}.$$

Assumptions of t-distribution

1. The parent population from which the sample is drawn is normal.
2. The sample observations are independent. (i.e.) the sample is random.
3. The population standard deviation σ is unknown.

Applications of t-distribution

The t-distribution has a wide number of applications in Statistics.

1. It is used to test if the sample mean $(\bar{x})$ differs significantly from the hypothetical value μ of the population mean.
2. It is used to test the significance of the difference between two sample means.
3. It is used to test the significance of an observed sample correlation coefficient and sample regression coefficient.
4. It is used to test the significance of observed partial and multiple correlation coefficients.

Note:

When $\gamma = 1$, Student's t-distribution reduces to Cauchy distribution. Therefore the pdf of Student's t-distribution is,

$$f(t) = \frac{1}{\sqrt{r}\,\beta\left(\frac{1}{2},\frac{\gamma}{2}\right)} \frac{1}{\left(1+\frac{t^2}{\gamma}\right)^{\frac{\gamma+1}{2}}};\ -\infty < t < \infty$$

When $\gamma = 1$,

$$\mathrm{f}(\mathrm{t}) = \frac{1}{\beta\left(\frac{1}{2},\frac{1}{2}\right)} \frac{1}{1+\mathrm{t}^2}; \quad -\infty < \mathrm{t} < \infty$$

$$= \frac{1}{\sqrt{\pi}\,\sqrt{\pi}} \frac{1}{1+t^2} \qquad \left(\because \beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\right)$$

$\mathrm{f}(\mathrm{t}) = \frac{1}{\pi} \frac{1}{1+\mathrm{t}^2}; \quad -\infty < \mathrm{t} < \infty$, where is the pdf of standard Cauchy distribution.

Hence, when $\gamma = 1$ Student's t-distribution reduced to standard Cauchy distribution.

5.4 F - Distribution

Definition

If X and Y are two independent χ^2 variates with n_1 *and* n_2 degrees of freedom respectively, then F-Statistic is defined by, $F = \frac{X/n_1}{Y/n_2}$.

In other words, F-distribution is defined as the ratio of two independent χ^2 variates divided by their respective degrees of freedom and it follows Snedecor's F-distribution with (n_1, n_2) degrees of freedom with probability function given by,

$$f(F)=\frac{\left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}} F^{\frac{n1}{2}-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n1+n2}{2}}}, \quad 0 \le F < \infty$$

Derivation of F-distribution:

Let $\chi_1^{\ 2}, \chi_2^{\ 2}$ be the two independent χ^2 variates with n_1 *and* n_2 degrees of freedom respectively, $\mathrm{f}(\chi_1^{\ 2}, \chi_2^{\ 2}) = \mathrm{f}(\chi_1^{\ 2})\ \mathrm{f}(\chi_2^{\ 2})$

$$=\frac{(\chi_1^{\ 2})^{\frac{n1}{2}-1}\ e^{-\frac{\chi_1^{\ 2}}{2}}}{2^{\frac{n1}{2}}\ \Gamma\frac{n_1}{2}}\ \frac{(\chi_2^{\ 2})^{\frac{n2}{2}-1}\ e^{-\frac{\chi_2^{\ 2}}{2}}}{2^{\frac{n2}{2}}\ \Gamma\frac{n_2}{2}}; \qquad 0 \le \chi_1^{\ 2} < \infty,\ 0 \le \chi_2^{\ 2} < \infty$$

$$=\frac{1}{2^{\frac{n1+n2}{2}}\ \Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}\ e^{-\frac{1}{2}\left(\chi_1^{2}+\chi_2^{2}\right)}\left(\chi_1^{\ 2}\right)^{\frac{n1}{2}-1}\left(\chi_2^{\ 2}\right)^{\frac{n2}{2}-1}$$

Put, $F=\frac{\chi_1^{\ 2}/n_1}{\chi_2^{\ 2}/n_2}$ *and* $Y=\chi_2^{\ 2}$

$$\Rightarrow \chi_1^{\ 2} = \mathrm{F}\frac{\chi_2^{\ 2}}{\mathrm{n}_2}\mathrm{n}_1$$

$$\therefore \chi_1^{\ 2} = \frac{n_1}{n_2}F\ y \quad and \quad \chi_2^{\ 2} = y$$

$$\therefore\ f(\chi_1^{\ 2}, \chi_2^{\ 2}) = \frac{1}{2^{\frac{n1+n2}{2}}\ \Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}\ e^{-\frac{y}{2}\left(\frac{n1}{n2}F+1\right)}\left(\frac{n_1}{n_2}F\ y\right)^{\frac{n1}{2}-1} y^{\frac{n_2}{2}-1}\ |J| \ \ for\ 0 \le F < \infty,\ 0 \le y < \infty$$

$$= \frac{1}{2^{\frac{n1+n2}{2}} \Gamma \frac{n_1}{2} \Gamma \frac{n_2}{2}} \, e^{-\frac{y}{2}\left(\frac{n1}{n2}F+1\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}-1} F^{\frac{n1}{2}-1} \, y^{\frac{n1+n2}{2}-2} \, |J|$$

$$\text{Where } J = \begin{vmatrix} \frac{\partial \chi_1^2}{\partial F} & \frac{\partial \chi_1^2}{\partial y} \\ \frac{\partial \chi_2^2}{\partial F} & \frac{\partial \chi_2^2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{n_1}{n_2} y & \frac{n_1}{n_2} F \\ 0 & 1 \end{vmatrix} = \frac{n_1}{n_2} y$$

$$\therefore f\left(\chi_1^2, \chi_2^2\right) = \frac{1}{2^{\frac{n1+n2}{2}} \Gamma \frac{n_1}{2} \Gamma \frac{n_2}{2}} \, e^{-\frac{y}{2}\left(\frac{n1}{n2}F+1\right)} F^{\frac{n1}{2}-1} \left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}-1} y^{\frac{n1+n2}{2}-2} \, \frac{n_1}{n_2} y$$

$$= \frac{1}{2^{\frac{n1+n2}{2}} \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} \, e^{-\frac{y}{2}\left(\frac{n1}{n2}F+1\right)} F^{\frac{n1}{2}-1} \left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}} y^{\frac{n1+n2}{2}-1}$$

$$\therefore f(F) = \int_0^\infty f\left(\chi_1^2, \chi_2^2\right) dy \qquad \left(\because f_X(x) = \int f(x, y)\, dy\right)$$

$$= \frac{1}{2^{\frac{n1+n2}{2}} \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} F^{\frac{n1}{2}-1} \left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}} \int_0^\infty e^{-\frac{y}{2}\left(\frac{n1}{n2}F+1\right)} y^{\frac{n1+n2}{2}-1} \, dy$$

$$\text{Put, } t = \frac{y}{2}\left(\frac{n_1}{n_2}F + 1\right) \Rightarrow y = \frac{2t}{\left(\frac{n_1}{n_2}F + 1\right)}$$

$$\therefore dt = \frac{1}{2}\left(\frac{n_1}{n_2}F + 1\right) dy \qquad \Rightarrow dy = \frac{2}{\frac{n_1}{n_2}F + 1} dt$$

$$\therefore f(F)=\frac{1}{2^{\frac{n1+n2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}F^{\frac{n1}{2}-1}\left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}}\int_0^{\infty}e^{-t}\left[\frac{2t}{\left(\frac{n_1}{n_2}F+1\right)}\right]^{\frac{n1+n2}{2}-1}\frac{2}{\left(\frac{n_1}{n_2}F+1\right)}dt$$

$$=\frac{1}{2^{\frac{n1+n2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}F^{\frac{n1}{2}-1}\left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}}2^{\frac{n1+n2}{2}}\frac{1}{\left(\frac{n_1}{n_2}F+1\right)^{\frac{n1+n2}{2}}}\int_0^{\infty}e^{-t}\,t^{\left(\frac{n1+n2}{2}\right)-1}dt$$

$$=\frac{1}{\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}F^{\frac{n1}{2}-1}\left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}}\frac{1}{\left(\frac{n_1}{n_2}F+1\right)^{\frac{n1+n2}{2}}}\Gamma\left(\frac{n_1+n_2}{2}\right)dt$$

$$=\frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\frac{F^{\frac{n1}{2}-1}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}}}{\left(\frac{n_1}{n_2}F+1\right)^{\frac{n1+n2}{2}}}$$ which is the probability function of F-distribution.

Constants of F-distribution r^{th} order moments about origin

By the definition of r^{th} moments about origin,

$$\mu_r{}'=E\left(X^r\right)=E\left[F^r\right]$$

$$=\int_0^{\infty}F^r\,f(F)\,dF$$

$$=\int_0^\infty F^r \frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \frac{F^{\frac{n1}{2}-1}\left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}}}{\left(\frac{n_1}{n_2}\ F+1\right)^{\frac{n1+n2}{2}}} dF$$

$$=\frac{\left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \int_0^\infty \frac{F^{\frac{n1}{2}+r-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n1+n2}{2}}} dF$$

Let $u=\frac{n_1}{n_2}F \Rightarrow F=\frac{n_2}{n_1}u \quad \Rightarrow dF=\frac{n_2}{n_1}du$

$$\mu_r{}' = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \int_0^\infty \frac{\left(\frac{n_2}{n_1}u\right)^{\frac{n1}{2}+r-1}}{(1+u)^{\frac{n1+n2}{2}}} \left(\frac{n_2}{n_1}\right)du$$

$$=\frac{\left(\frac{n_1}{n_2}\right)^{\frac{n1}{2}}\left(\frac{n_2}{n_1}\right)^{\frac{n1}{2}+r}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \int_0^\infty \frac{u^{\frac{n1}{2}+r-1}}{(1+u)^{\frac{n1+n2}{2}}} du$$

We know that

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx=\beta(m,n)$$

$$= \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \beta\left(\frac{n_1}{2} + r, \frac{n_2}{2} - r\right)$$

$$\therefore \mu_r' = \left(\frac{n_2}{n_1}\right)^r \frac{\beta\left(\frac{n_1}{2} + r, \frac{n_2}{2} - r\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \qquad (1)$$

Put r=1 in equation (1),

$$\mu_1' = \left(\frac{n_2}{n_1}\right) \frac{\beta\left(\frac{n_1}{2} + 1, \frac{n_2}{2} - 1\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$$

$$= \frac{n_2}{n_1} \frac{\Gamma\left(\frac{n_1}{2}\right) + 1\, \Gamma\left(\frac{n_2}{2}\right) - 1}{\Gamma\left(\frac{n_1 + n_2}{2}\right)} \frac{\Gamma\left(\frac{n_1}{2} + \frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{n_2}{n_1} \frac{\frac{n_1}{2}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2} - 1\right)}{\Gamma \frac{n_1}{2}\left(\frac{n_2}{2} - 1\right)\Gamma\left(\frac{n_2}{2} - 1\right)}$$

$$= \frac{n_2}{n_1} \frac{\frac{n_1}{2}}{\frac{n_2 - 2}{2}}$$

$$\therefore \mu_1' = \frac{n_2}{n_2 - 2}.$$

Put r=2 in equation (1), we get,

$$\mu_2' = \left(\frac{n_2}{n_1}\right)^2 \frac{\beta\left(\frac{n_1}{2}+2, \frac{n_2}{2}-2\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\Gamma\left(\frac{n_1}{2}+2\right)\Gamma\left(\frac{n_2}{2}-2\right)}{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\Gamma\left(\frac{n_1}{2}\right)+2\Gamma\left(\frac{n_2}{2}-2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\left(\frac{n_1}{2}+1\right)!\ \left(\frac{n_2}{2}-3\right)!}{\left(\frac{n_1}{2}-1\right)!\ \left(\frac{n_2}{2}-1\right)!}$$

$$= \frac{{n_2}^2}{{n_1}^2} \frac{\left(\frac{n_1}{2}+1\right)\left(\frac{n_1}{2}\right)\left(\frac{n_1}{2}-1\right)!\ \left(\frac{n_2}{2}-3\right)!}{\left(\frac{n_1}{2}-1\right)!\left(\frac{n_2}{2}-1\right)\left(\frac{n_2}{2}-2\right)\left(\frac{n_2-3}{2}\right)!}$$

$$= \frac{{n_2}^2}{{n_1}^2} \frac{\left(\frac{n_1}{2}+1\right)\left(\frac{n_1}{2}\right)}{\left(\frac{n_2}{2}-1\right)\left(\frac{n_2}{2}-2\right)}$$

$$\mu_2' = \frac{{n_2}^2}{n_1} \frac{(n_1+2)\ /\ 4}{(n_2-2)(n_2-4)}.$$

$$\mathrm{Var}(X) = \mu_2' - \left(\mu_1'\right)^2$$

$$=\frac{n_2^{\,2}}{n_1}\frac{(n_1+2)\,/\,4}{(n_2-2)(n_2-4)}-\left(\frac{n_2}{n_2-2}\right)^2$$

$$=\frac{(n_2-2)n_2^{\,2}(n_1+2)-n_2^{\,2}(n_2-4)n_1}{n_1(n_2-2)^2(n_2-4)}$$

$$=\frac{n_2^{\,3}n_1+2n_2^{\,3}-2n_1n_2^{\,2}-4n_2^{\,2}-n_2^{\,3}n_1+4n_2^{\,2}n_1}{n_1(n_2-2)^2(n_2-4)}$$

$$=\frac{2n_2^{\,2}(-n_1+n_2-2+2n_1)}{n_1(n_2-2)^2(n_2-4)}$$

$$\mathrm{Var}(X)=\frac{2n_2^{\,2}(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}\ .$$

Exercise

1. Define central χ^2 distribution.
2. Give the applications of student's t-distribution.
3. Define F distribution.
4. Prove that limiting case of t-distribution is a normal distribution.
5. Define central χ^2 distribution and derive its moment generating function cumulants constants.
6. Prove that limiting form of χ^2 distribution is a normal distribution
7. Derive the probability density function of student's t-distribution.
8. Find the constant of t-distribution.
9. State applications of Chi-square distribution.
10. Derive the constants of central F distribution.
11. Derive the probability function of central F distribution.

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